9.7 AP Practice Problems (p.696) – Integrals of Vector Functions & Projectile Motion

1. Find $\int_0^1 \mathbf{r}(t)\,dt$ where $\mathbf{r}(t) = \left\langle \frac{1}{t^2+1}, \frac{1}{t+1} \right\rangle$.

   (A) $\frac{\pi}{4} + \ln 2$ $\quad$ (B) $\boxed{\left\langle \frac{\pi}{4}, \ln 2 \right\rangle}$

   (C) $\left\langle -\frac{1}{2}, \ln 2 \right\rangle$ $\quad$ (D) $\langle \tan^{-1} t, \ln(t+1) \rangle$

$\left\langle \int_0^1 \frac{1}{t^2+1}\,dt, \int_0^1 \frac{1}{t+1}\,dt \right\rangle$

$\frac{1}{1}\arctan(t)\Big]_0^1, \ln|t+1|\Big]_0^1$

$\arctan(1) - \arctan 0, \ln(2) - \ln(0)$

$\boxed{\langle \pi/4, \ln 2 \rangle}$

2. $\int [\langle \sec^2 t, \cos t \rangle]\,dt =$

   (A) $\langle \tan t, \sin t \rangle$ $\quad$ (B) $\langle \tan t, -\sin t \rangle + c$

   (C) $\boxed{\langle \tan t, \sin t \rangle + c}$ $\quad$ (D) $\left\langle \frac{\sec^3 t}{3}, \sin t \right\rangle + c$

$\langle \tan t + C_1, \sin(t) + C_2 \rangle$

3. A particle is moving with velocity

$$\mathbf{v}(t) = \langle \pi\cos(\pi t), 3t^2 + 1 \rangle \text{ m/s}$$

for $0 \le t \le 10$ seconds. Given that the position of the particle at time $t = 2$ s is $\mathbf{r}(2) = \langle 3, -2 \rangle$, the position vector of the particle at t is

   (A) $\langle 3, -12 \rangle$ $\quad$ (B) $\langle 3 + \sin(\pi t), t^3 + t + 10 \rangle$

   (C) $\langle \sin(\pi t), t^3 + t \rangle$ $\quad$ (D) $\boxed{\langle 3 + \sin(\pi t), t^3 + t - 12 \rangle}$

$r(t) = \int \langle \pi\cos(\pi t), 3t^2+1 \rangle\,dt$

$u = \pi t$

$\frac{du}{dt} = \pi$

$dt = \frac{du}{\pi}$

$\langle \sin(\pi t) + C_1, \frac{3t^3}{3} + t + C_2 \rangle$

$r(2) = \langle 3, -2 \rangle$

$\sin(\pi t) + C_1 = 3$ at $t = 2$

$\sin(2\pi) + C_1 = 3 \rightarrow C_1 = 3$

$\frac{3t^3}{3} + t + C_2 = -2$ at $t = 2$

$2^3 + 2 + C_2 = -2 \rightarrow C_2 = -12$

$\boxed{r(t) = \langle \sin(\pi t) + 3, t^3 + t - 12 \rangle}$

4. The solution to the vector differential equation $\mathbf{r}'(t) = \langle 4e^{4t}, 3t^2 \rangle$ given $\mathbf{r}(0) = \langle 2, -1 \rangle$ is

(A) $\langle 1 + e^{4t}, -t^3 \rangle$ $\boxed{\text{(B)}}$ $\langle 1 + e^{4t}, t^3 - 1 \rangle$

(C) $\langle 2 + e^{4t}, t^3 - 1 \rangle$ (D) $\langle 3 + e^{4t}, t^3 - 1 \rangle$

$\boxed{r(t) = \langle e^{4t}+1, t^3-1 \rangle}$

$r(t) = \int \langle 4e^{4t}, 3t^2 \rangle dt$

$\langle e^{4t}+C_1, \frac{3t^3}{3}+C_2 \rangle$

$u = 4t$

$\frac{du}{dt} = 4$

$dt = \frac{du}{4}$

$\int 4e^{u} \cdot \frac{du}{4} = e^u \to e^{4t}$

$r(0) = \langle 2, -1 \rangle$

$e^{4(0)} + C_1 = 2$

$C_1 = 1$

$t^3 + C_2 = -1$

$(0)^3 + C_2 = -1 \to C_2 = -1$

5. If an object travels in the xy-plane along the curve traced out by the vector function $\mathbf{r}(t) = \langle t^{3/2}, -t \rangle$ for $t \geq 0$, then the total distance traveled by the object from $t = 0$ to $t = 4$ is

(A) $\frac{16}{3}$ (B) $\frac{2}{3}10^{3/2}$ (C) $10^{3/2} - 1$ $\boxed{\text{(D)}}$ $\frac{8}{27}[10^{3/2} - 1]$

$u = 9t+4$ $\frac{du}{dt} = 9$ $dt = \frac{du}{9}$

$\frac{1}{2}\int u^{1/2} \cdot \frac{du}{9} \to \frac{1}{18}\int u^{1/2} du$

$\frac{1}{18} \cdot \frac{u^{3/2}}{3/2} \to \frac{2}{3} \cdot \frac{1}{18} u^{3/2}$

$\int_0^4 |r'(t)| dt \to \int_0^4 \sqrt{\left[\frac{3}{2}t^{1/2}\right]^2 + [-1]^2} dt$

$r'(t) = \langle \frac{3}{2}t^{1/2}, -1 \rangle$ $\int_0^4 \sqrt{\frac{9}{4}t + 1}\, dt \to$

$\int \sqrt{\frac{9t+4}{4}}$

$\frac{1}{2}\int (9t+4)^{1/2} dt$

$\frac{1}{27}(9t+4)^{3/2}\Big]_0^4$

$\boxed{\frac{1}{27}(40)^{3/2} - \frac{1}{27}(4)^{3/2} \quad \frac{8}{27}\left[10^{3/2} - 1\right]}$

6. A flare is launched at an angle of elevation 60° ($\frac{\pi}{3}$ radians) with initial speed $\|\mathbf{v}(0)\| = 200$ ft/s from a stationary barge's deck which is three feet above the water's surface. The only external force acting on the flare is gravity, so $\mathbf{a}(t) = \langle 0, -32 \rangle$ ft/s^2.

$\boxed{\text{Skip}}$

(a) Find the velocity vector $\mathbf{v} = \mathbf{v}(t)$ of the flare. $\langle 100, 100\sqrt{3} - 32t \rangle$

(b) Find the position vector $\mathbf{r} = \mathbf{r}(t)$ of the flare. $\langle 100t, 3 + 100t\sqrt{3} - 16t^2 \rangle$