BC Calculus 9.7-9.10 Review WS #2

find a power series for the given function, centered at the given value of *a*. Give the first four nonzero terms and the general term

1) $f(x) = \dfrac{1}{1+x^2}$, $a = 0$

2) Let f be the function given by $f(t) = \dfrac{4}{1+t^2}$ and G be the function given by $G(x) = \int_0^x f(t)dt$.

(a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.

(b) Find the first four nonzero terms and the general term of the power series expansion of $G(x)$ about $x = 0$.

(c) Find the interval of convergence of the power series in part (b). Justify your answer.

3) Let f be the function given by $f(x) = e^{-2x^2}$

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

(b) Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

(c) (Calculator Permitted) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x = 0$. Show that $|f(x) - g(x)| < 0.02$ for $-0.6 \le x \le 0.6$.

4) The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

(b) For what values of x does the given series converge? Show your reasoning.

(c) Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

Key

find a power series for the given function, centered at the given value of a. Give the first four nonzero terms and the general term (3 options: a) Taylor's Rule b) Long Division c) Geometric Series $S=\frac{a_1}{1-r}$

1) $f(x)=\frac{1}{1+x^2}$, $a=0$ $=\frac{1}{1-(-x^2)}$

$= 1-x^2+x^4-x^6+\dots+(-1)^n(x)^{2n}+\dots$

2) Let f be the function given by $f(t)=\frac{4}{1+t^2}$ and G be the function given by $G(x)=\int_0^x f(t)dt$.

(a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t=0$.

$4\left(\frac{1}{1+t^2}\right)=4(1-t^2+t^4-t^6+\dots)=4-4t^2+4t^4-4t^6+\dots(-1)^n\cdot 4t^{2n}$

(b) Find the first four nonzero terms and the general term of the power series expansion of $G(x)$ about $x=0$.

$\int_0^x f(t)\,dt = 4t-\frac{4t^3}{3}+\frac{4t^5}{5}-\frac{4t^7}{7}\Big]_0^x = 4x-\frac{4x^3}{3}+\frac{4x^5}{5}-\frac{4x^7}{7}+\dots+\frac{(-1)^n 4x^{2n+1}}{2n+1}$

(c) Find the interval of convergence of the power series in part (b). Justify your answer.

* Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

$\lim_{n\to\infty}\left|\frac{4x^{2(n+1)+1}}{2(n+1)+1}\cdot\frac{2n+1}{4x^{2n+1}}\right|$

$=\lim_{n\to\infty}\left|\frac{x^{2n+3}\cdot(2n+1)}{x^{2n+1}\cdot(2n+3)}\right|=|x^2|<1$ $\quad -1<x<1$

Endpts do not converge (Geometric series)

3) Let f be the function given by $f(x)=e^{-2x^2}$

(a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x=0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{-2x^2} = 1 - 2x^2 + \frac{2^2x^4}{2!} - \frac{2^3x^6}{3!} + \dots + \frac{(-1)^n 2^n x^{2n}}{n!}$$

(b) Find the interval of convergence of the power series for $f(x)$ about $x=0$. Show the analysis that leads to your conclusion.

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{2^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{2x^2}{n+1}\right| = 0 < 1 \text{ for all } x$$

I.O.C. is $(-\infty, \infty)$

(c) (Calculator Permitted) Let g be the function given by the sum of the first four nonzero terms of the power series for $f(x)$ about $x=0$. Show that $|f(x)-g(x)|<0.02$ for $-0.6 \le x \le 0.6$.

Alt. series Remainder. $|f(x)-g(x)| < \left|\frac{2^4x^8}{4!}\right|$ ← 1st unused term $= \frac{16x^8}{4\cdot3\cdot2} = \left|\frac{2}{3}x^8\right|$

$=\left|\frac{2}{3}x^8\right|$ is maximized on $-0.6 \le x \le 0.6$ when $x=0.6$ or $x=-0.6$

$\left|\frac{2}{3}(0.6)^8\right| = 0.01119744 < 0.02$

4) The Maclaurin series for $f(x)$ is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\dots+\frac{x^n}{(n+1)!}+\dots$

(a) Find $f'(0)$ and $f^{(17)}(0)$.

$\frac{f^{(n+1)}(c)}{n!}(x-c)^n$

$$\frac{f'(0)}{1!}(x-0)^1 = \frac{x}{2} \quad \text{so } f'(0)x = \frac{x}{2}$$

$f'(0) = \frac{1}{2}$

$$\frac{f^{(17)}(0)}{17!}(x-0)^{17} = \frac{x^{17}}{18!}$$

$$\frac{f^{(17)}(0)}{17!}x^{17} = \frac{x^{17}}{18!} \rightarrow f^{(17)}(0) = \frac{1}{18}$$

(b) For what values of x does the given series converge? Show your reasoning.

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+2)!}\cdot\frac{(n+1)!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+2}\right| = 0 < 1$$

I.O.C: $(-\infty, \infty)$

(c) Let $g(x)=xf(x)$. Write the Maclaurin series for $g(x)$ in terms of a familiar function without using series. Then, write $f(x)$ in terms of the same familiar function.

$$g(x) = x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\dots\right)$$

$$g(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = e^x - 1$$

$$g(x) = xf(x)$$

$$e^x - 1 = xf(x)$$

$$\frac{e^x-1}{x} = f(x)$$