BC Calculus 9.7-9.10 Review WS #4

1) Taylor Rule

2) Lagrange Error Bound

3) MacLaurin Series (first 4 terms and general rule) for:

a. sin x

b. cos x

c. e^x

4. Find a power series for the function, centered at c, and determine the interval of convergence

$f(x) = \frac{3}{2x-1}, \quad c = 2$

$f(x) = \frac{4}{5-x}, \quad c = -2$

5. Find the interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$$

6.

Let f be the function given by $f(x) = \cos\left(3x + \frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right) - P\left(\frac{1}{6}\right)\right| < \frac{1}{3000}$.

c) Find the value for $f^{(68)}(0)$

d) Find $f^{(57)}(0)$

7. Find the Maclaurin Series for the function (first 4 terms and general rule)

a) $f(x) = \cos x^{3/2}$

b) $g(x) = 2\sin x^3$

c) $f(x) = e^{3x} + e^{-3x}$

d) $g(x) = \frac{\cos(4x)}{3\sqrt{x}}$

8.

Which of the following is the coefficient of x^4 in the Maclaurin polynomial generated by $\cos(3x)$?

(A) $\frac{27}{8}$ (B) 9 (C) $\frac{1}{24}$ (D) 0 (E) $-\frac{27}{8}$

b. Find the coefficient of x^{68} in the Maclaurin polynomial generated by cos(3x)?

9. Find the Taylor polynomial of degree n centered at x = c for the below:

a) f(x) = ln x , n = 5 , c = 1

b) g(x) = $\frac{1}{x}$, n = 5, c = 1

Key

BC Calculus **9.7-9.10** **Review WS #4**

Write down:

1) Taylor Rule

$$\frac{f^{n}(c)}{n!}(x-c)^{n}$$

2) Lagrange Error Bound

$$R_n(x) = \left|\frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

3) MacLaurin Series (first 4 terms and general rule) for:

a. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$

b. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$

c. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

4. Find a power series for the function, centered at c, and determine the interval of convergence

$f(x) = \frac{3}{2x-1}$, $c = 2$

$$\frac{3}{2(x-2)-1+4} = \frac{3}{3+2(x-2)}$$

$$\frac{\frac{3}{3}}{\frac{3}{3}-\left[\frac{-2}{3}(x-2)\right]} = \frac{1}{1-\left[\frac{-2}{3}(x-2)\right]} = \sum_{n=0}^{\infty} 1\left(\frac{-2}{3}(x-2)\right)^n$$

$$\left|\frac{2}{3}(x-2)\right| < 1 \qquad |x-2| < \frac{3}{2} \qquad \boxed{\frac{1}{2} < x < \frac{7}{2}}$$

$f(x) = \frac{4}{5-x}$, $c = -2$

$$\frac{4}{2+5-(x+2)}$$

$$\frac{4}{7-(x+2)} = \frac{\frac{4}{7}}{\frac{7}{7}-\frac{1}{7}(x+2)}$$

$$\sum_{n=0}^{\infty} \frac{4}{7}\left[\frac{1}{7}(x+2)\right]^n \qquad \frac{1}{7}(x+2) < 1 \qquad |x+2| < 7 \qquad \boxed{-9 < x < 5}$$

5. Find the interval of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$$

$$\lim_{n\to\infty}\left|\frac{3^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{3^n(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3(x-2)\cdot n}{n+1}\right| = 3|x-2| < 1$$

$$|x-2| < \frac{1}{3}$$

center: $c = 2$

radius $= \frac{1}{3}$

$[(\frac{5}{3}, \frac{7}{3})]$

test $x = \frac{5}{3}$

$$\sum \frac{3^n(-\frac{1}{3})^n}{n} = \frac{(-1)^n}{n}$$ converges by AST

test $x = \frac{7}{3}$

$$\sum \frac{3^n(\frac{1}{3})^n}{n}$$ diverges by harmonic series

$$\boxed{\text{I.O.C. } [\tfrac{5}{3}, \tfrac{7}{3})}$$

6.

Let f be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

$f(x)=\cos(3x+\frac{\pi}{6})$ $\quad f(0)=\frac{\sqrt{3}}{2}$

R(1) $f'(x)=-3\sin(3x+\pi/6)$ $\quad f'(0)=\frac{-3}{2}$

R(2) $f''(x)=-9\cos(3x+\pi/6)$ $\quad f''(0)=-\frac{9\sqrt{3}}{2}=\frac{-3^2\sqrt{3}}{2}$

R(3) $f'''(x)=27\sin(3x+\pi/6)$ $\quad f'''(0)=\frac{27}{2}=\frac{3^3}{2}$

R(0) $f^{4}(x)=81\cos(3x+\pi/6)$ $\quad f^4(0)=\frac{81\sqrt{3}}{2}=\frac{3^4\sqrt{3}}{2}$

$f^5(x)=-243\sin(3x+\pi/6)$ $\quad f^5(0)=\frac{-243}{2}=\frac{-3^5}{2}$

$f(x)=P_4(x)=$

$$\boxed{\frac{\sqrt{3}}{2}-\frac{3}{2}x-\frac{9\sqrt{3}}{4}x^2+\frac{9}{4}x^3+\frac{27\sqrt{3}}{16}x^4}$$

$$P_4(x)=\frac{\sqrt{3}}{2}-\frac{3}{2}(x-0)-\frac{\frac{9\sqrt{3}}{2}}{2!}(x-0)^2+\frac{\frac{27}{2}}{3!}(x-0)^3+\frac{\frac{81\sqrt{3}}{2}}{4!}(x-0)^4$$

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

$$R_n(x)=\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

$x=1/6$ $\quad c=0$

$$=\left|\frac{f^{(5)}(z)}{5!}(1/6-0)^5\right|\le\left|\frac{243}{5!}\left(\frac{1}{6}\right)^5\right|=\boxed{0.0002604<\frac{1}{3000}}$$

c) Find the value for $f^{(68)}(0)$

$4\overline{)68}$ = 17 R0; 4, 28

$$\frac{f^{68}(0)}{68!}(x-0)^{68}=\frac{\frac{3^{68}\sqrt{3}}{2}}{68!}(x)^{68}$$

$$\boxed{f^{68}(0)=\frac{3^{68}\sqrt{3}}{2}}$$

$f^{(57)}(0)$

$4\overline{)57}$ = 14 R1; 4, 17, -16, 1

$$\boxed{\frac{-3^{57}}{2}}$$

7. Find the Maclaurin Series for the function (first 4 terms and general rule) (9.10) 25, 26, 28

a) $f(x) = \cos x^{3/2}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$

$\cos(x^{3/2}) = \frac{(-1)^n (x^{3/2})^{2n}}{(2n)!} = \boxed{\frac{(-1)^n x^{3n}}{(2n)!}} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{(2n)!}$

$\boxed{\cos x^{3/2} = 1 - \frac{x^3}{2!} + \frac{x^6}{4!} - \frac{x^9}{6!} + \dots \frac{(-1)^n x^{3n}}{(2n)!}}$

b) $g(x) = 2\sin x^3$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$

$2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$

$\sin x^3 = \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

$\boxed{2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!} = 2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \frac{2x^{21}}{7!}}$

c) $f(x) = e^{3x} + e^{-3x}$

$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!}$

$e^{-3x} = 1 - 3x + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} +$

$\boxed{= 2\sum_{n=0}^{\infty} \frac{(3x)^{2n}}{(2n)!} = 2 + \frac{2(3x)^2}{2!} + \frac{2(3x)^4}{4!} + \frac{2(3x)^6}{6!}}$

d) $g(x) = \frac{\cos(4x)}{3\sqrt{x}}$

$\boxed{\frac{1}{3}\sum_{n=0}^{\infty} \frac{(4)^{2n} x^{2n-1/2}}{(2n)!}}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos(4x) = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!}$

$\frac{\cos(4x)}{3\sqrt{x}} = \frac{1}{3\sqrt{x}} - \frac{(4x)^2}{3\sqrt{x}\cdot 2} + \frac{(4x)^4}{3\sqrt{x}\cdot 4!} - \frac{(4x)^6}{3\sqrt{x}\cdot 6!}$

8.

Which of the following is the coefficient of x^4 in the Maclaurin polynomial generated by $\cos(3x)$?

(A) $\frac{27}{8}$ (B) 9 (C) $\frac{1}{24}$ (D) 0 (E) $-\frac{27}{8}$

$f = \cos(3x)$ R0

$f' = -3\sin(3x)$ R1

$f'' = -3^2\cos(3x)$ R2

$f''' = 3^3\sin(3x)$ R3

$f^{(4)} = 3^4\cos(3x)$

$\frac{f^{(4)}(0)}{4!}(x-0)^4 = \frac{3^4(1)}{4!}x^4 = \frac{81}{24}x^4 = \boxed{\frac{27}{8}}$

b. Find the coefficient of x^{68} in the Maclaurin polynomial generated by cos(3x)?

$\frac{f^{(68)}(0)}{68!}(x-0)^{68}$ $\frac{3^{68}(1)}{68!}x^{68} = \boxed{\frac{3^{68}}{68!}}$

$f^{68} = 3^{68}\cos(3x)$

$68 \div 4 = 17$ $\boxed{R0}$

9. Find the Taylor polynomial of degree n centered at x = c for the below:

a) f(x) = ln x , n = 5 , c = 1 $f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $f'(1) = 1$

$f''(x) = -1x^{-2}$ $f''(1) = -1$

$f'''(x) = 2x^{-3}$ $f'''(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $f^{(4)}(1) = -6$

$f^{(5)}(x) = 24x^{-5}$ $f^{(5)}(1) = 24$

$P_5(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5$

b) g(x) = $\frac{1}{x}$, n = 5, c = 1

$f(x) = \frac{1}{x}$ $f(1) = 1$

$f' = -1x^{-2}$ $f'(1) = -1$

$f'' = 2x^{-3}$ $f''(1) = 2$

$f''' = -6x^{-4}$ $f'''(1) = -6$

$f^{(4)} = 24x^{-5}$ $f^{4}(1) = 24$

$f^{5} = -120x^{-6}$ $f^{5}(1) = -120$

$P_5(x) = 1 - 1(x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4 - \frac{120}{5!}(x-1)^5$

$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$