108. For $n = 1, 2, 3, \ldots, -|a_n| \le a_n \le |a_n| \Rightarrow -\sum_{n=1}^{k}|a_n| \le \sum_{n=1}^{k} a_n \le \sum_{n=1}^{k}|a_n|$.

Taking limits as $k \to \infty$, $-\sum_{n=1}^{\infty}|a_n| \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty}|a_n| \Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty}|a_n|$.

109. First prove Abel's Summation Theorem:

If the partial sums of $\sum a_n$ are bounded and if $\{b_n\}$ decreases to zero, then $\sum a_n b_n$ converges.

Let $S_k = \sum_{i=1}^{k} a_i$. Let M be a bound for $\{|S_k|\}$.

$$\begin{aligned} a_1b_1 + a_2b_2 + \cdots + a_nb_n &= S_1b_1 + (S_2 - S_1)b_2 + \cdots + (S_n - S_{n-1})b_n \\ &= S_1(b_1 - b_2) + S_2(b_2 - b_3) + \cdots + S_{n-1}(b_{n-1} - b_n) + S_nb_n \\ &= \sum_{i=1}^{n-1} S_i(b_i - b_{i+1}) + S_nb_n \end{aligned}$$

The series $\sum_{i=1}^{\infty} S_i(b_i - b_{i+1})$ is absolutely convergent because $|S_i(b_i - b_{i+1})| \le M(b_i - b_{i+1})$ and $\sum_{i=1}^{\infty}(b_i - b_{i+1})$ converges to b_1.

Also, $\lim_{n\to\infty} S_nb_n = 0$ because $\{S_n\}$ bounded and $b_n \to 0$. Thus, $\sum_{n=1}^{\infty} a_nb_n = \lim_{n\to\infty}\sum_{i=1}^{n} a_ib_i$ converges.

Now let $b_n = \frac{1}{n}$ to finish the problem.

110. Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n \Big/ \frac{(n-1)!}{n^{n-1}}\left(\frac{19}{7}\right)^{n-1}\right] = \lim_{n\to\infty}\left[\frac{n \cdot n^{n-1}}{(n+1)^n}\left(\frac{19}{7}\right)\right] = \lim_{n\to\infty}\left[\frac{1}{(1 + (1/n))^n}\left(\frac{19}{7}\right)\right] = \frac{19}{7}\cdot\frac{1}{e} < 1$$

So, the series converges.

Section 9.7 Taylor Polynomials and Approximations

1. $y = -\frac{1}{2}x^2 + 1$

Parabola

Matches (d)

2. $y = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

y-axis symmetry

Three relative extrema

Matches (c)

3. $y = e^{-1/2}[(x + 1) + 1]$

Linear

Matches (a)

4. $y = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

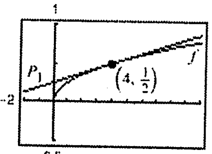
Cubic

Matches (b)

5. $f(x) = \frac{\sqrt{x}}{4}$, $C = 4$, $f(4) = \frac{1}{2}$

$f'(x) = \frac{1}{8\sqrt{x}}$, $f'(4) = \frac{1}{16}$

$$\begin{aligned} P_1(x) &= f(4) + f'(4)(x - 4) \\ &= \frac{1}{2} + \frac{1}{16}(x - 4) \\ &= \frac{1}{16}x + \frac{1}{4} \end{aligned}$$



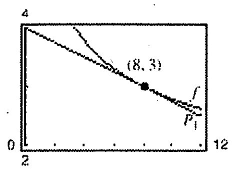
P_1 is the first-degree Taylor polynomial for f at 4.

6. $f(x) = \dfrac{6}{\sqrt[3]{x}} = 6x^{-1/3}$ $\quad f(8) = 3$

$f'(x) = -2x^{-4/3}$ $\quad f'(8) = -\dfrac{1}{8}$

$P_1(x) = f(8) + f'(8)(x - 8)$

$= 3 - \dfrac{1}{8}(x - 8) = -\dfrac{1}{8}x + 4$



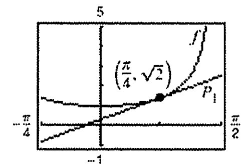
P_1 is the first degree Taylor polynomial for f at 8.

7. $f(x) = \sec x$ $\quad f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x$ $\quad f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$P_1(x) = f\left(\dfrac{\pi}{4}\right) + f'\left(\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right)$

$P_1(x) = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right)$



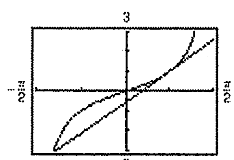
P_1 is called the first degree Taylor polynomial for f at $\dfrac{\pi}{4}$.

8. $f(x) = \tan x$ $\quad f\left(\dfrac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\quad f'\left(\dfrac{\pi}{4}\right) = 2$

$P_1 = f\left(\dfrac{\pi}{4}\right) + f'\left(\dfrac{\pi}{4}\right)\left(x - \dfrac{\pi}{4}\right) = 1 + 2\left(x - \dfrac{\pi}{4}\right)$

$P_1(x) = 2x + 1 - \dfrac{\pi}{2}$



P_1 is called the first degree Taylor polynomial for f at $\dfrac{\pi}{4}$.

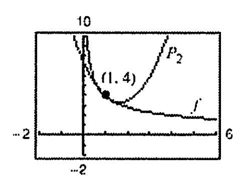
9. $f(x) = \dfrac{4}{\sqrt{x}} = 4x^{-1/2}$ $\quad f(1) = 4$

$f'(x) = -2x^{-3/2}$ $\quad f'(1) = -2$

$f''(x) = 3x^{-5/2}$ $\quad f''(1) = 3$

$P_2 = f(1) + f'(1)(x - 1) + \dfrac{f''(1)}{2}(x - 1)^2$

$= 4 - 2(x - 1) + \dfrac{3}{2}(x - 1)^2$



x	0	0.8	0.9	1.0	1.1	1.2	2
$f(x)$	Error	4.4721	4.2164	4.0	3.8139	3.6515	2.8284
$P_2(x)$	7.5	4.46	4.215	4.0	3.815	3.66	3.5

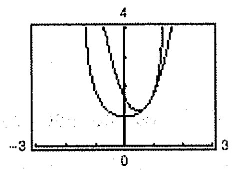
10. $f(x) = \sec x \qquad f\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f'(x) = \sec x \tan x \qquad f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

$f''(x) = \sec^3 x + \sec x \tan^2 x \qquad f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\pi/4)}{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$



x	-2.15	0.585	0.685	$\pi/4$	0.885	0.985	1.785
$f(x)$	-1.8270	1.1995	1.2913	1.4142	1.5791	1.8088	-4.7043
$P_2(x)$	15.5414	1.2160	1.2936	1.4142	1.5761	1.7810	4.9475

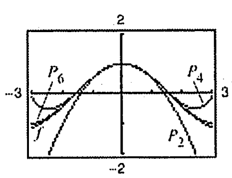
11. $f(x) = \cos x$

$P_2(x) = 1 - \frac{1}{2}x^2$

$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$

(a)



(b) $f'(x) = -\sin x \qquad P_2'(x) = -x$

$f''(x) = -\cos x \qquad P_2''(x) = -1$

$f''(0) = P_2''(0) = -1$

$f'''(x) = \sin x \qquad P_4'''(x) = x$

$f^{(4)}(x) = \cos x \qquad P_4^{(4)}(x) = 1$

$f^{(4)}(0) = 1 = P_4^{(4)}(0)$

$f^{(5)}(x) = -\sin x \qquad P_6^{(5)}(x) = -x$

$f^{(6)}(x) = -\cos x \qquad P^{(6)}(x) = -1$

$f^{(6)}(0) = -1 = P_6^{(6)}(0)$

(c) In general, $f^{(n)}(0) = P_n^{(n)}(0)$ for all n.

12. $f(x) = x^2 e^x,\ f(0) = 0$

(a) $f'(x) = (x^2 + 2x)e^x \qquad f'(0) = 0$

$f''(x) = (x^2 + 4x + 2)e^x \qquad f''(0) = 2$

$f'''(x) = (x^2 + 6x + 6)e^x \qquad f'''(0) = 6$

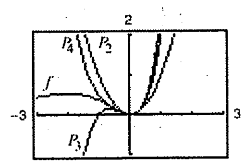
$f^{(4)}(x) = (x^2 + 8x + 12)e^x \qquad f^{(4)}(0) = 12$

$P_2(x) = \frac{2x^2}{2!} = x^2$

$P_3(x) = x^2 + \frac{6x^3}{3!} = x^2 + x^3$

$P_4(x) = x^2 + x^3 + \frac{12x^4}{4!} = x^2 + x^3 + \frac{x^4}{2}$

(b)



(c) $f''(0) = 2 = P_2''(0)$

$f'''(0) = 6 = P_3'''(0)$

$f^{(4)}(0) = 12 = P_4^{(4)}(0)$

(d) $f^{(n)}(0) = P_n^{(n)}(0)$

13. $f(x) = e^{4x}$ $\quad f(0) = 1$

$f'(x) = 4e^{4x}$ $\quad f'(0) = 4$

$f''(x) = 16e^{4x}$ $\quad f''(0) = 16$

$f'''(x) = 64e^{4x}$ $\quad f'''(0) = 64$

$f^{(4)}(x) = 256e^{4x}$ $\quad f^{(4)}(0) = 256$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4$$

14. $f(x) = e^{-x}$ $\quad f(0) = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = 1$

$f'''(x) = -e^{-x}$ $\quad f'''(0) = -1$

$f^{(4)}(x) = e^{-x}$ $\quad f^{(4)}(0) = 1$

$f^{(5)}(x) = -e^{-x}$ $\quad f^{(5)}(0) = -1$

$$P_5(x) = f(0) + f'(0)x + \frac{f'(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$+ \frac{f^{(5)}(0)}{5!}x^5 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

15. $f(x) = e^{-x/2}$ $\quad f(0) = 1$

$f'(x) = -\frac{1}{2}e^{-x/2}$ $\quad f'(0) = -\frac{1}{2}$

$f''(x) = \frac{1}{4}e^{-x/2}$ $\quad f''(0) = \frac{1}{4}$

$f'''(x) = -\frac{1}{8}e^{-x/2}$ $\quad f'''(0) = -\frac{1}{8}$

$f^{(4)}(x) = \frac{1}{16}e^{-x/2}$ $\quad f^{(4)}(0) = \frac{1}{16}$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{1}{384}x^4$$

16. $f(x) = e^{x/3}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{3}e^{x/3}$ $\quad f'(0) = \frac{1}{3}$

$f''(x) = \frac{1}{9}e^{x/3}$ $\quad f''(0) = \frac{1}{9}$

$f'''(x) = \frac{1}{27}e^{x/3}$ $\quad f'''(0) = \frac{1}{27}$

$f^{(4)}(x) = \frac{1}{81}e^{x/3}$ $\quad f^{(4)}(x) = \frac{1}{81}$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 + \frac{1}{3}x + \frac{1/9}{2!}x^2 + \frac{1/27}{3!}x^3 + \frac{1/81}{4!}x^4$$
$$= 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4$$

17. $f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}(0) = 1$

$$P_5(x) = 0 + (1)x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$
$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

18. $f(x) = \cos \pi x$ $\quad f(0) = 1$

$f'(x) = -\pi \sin \pi x$ $\quad f'(0) = 0$

$f''(x) = -\pi^2 \cos \pi x$ $\quad f''(0) = -\pi^2$

$f'''(x) = \pi^3 \sin \pi x$ $\quad f'''(0) = 0$

$f^{(4)}(x) = \pi^4 \cos \pi x$ $\quad f^{(4)}(0) = \pi^4$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$
$$= 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4$$

19. $f(x) = xe^x$ $\quad f(0) = 0$

$f'(x) = xe^x + e^x$ $\quad f'(0) = 1$

$f''(x) = xe^x + 2e^x$ $\quad f''(0) = 2$

$f'''(x) = xe^x + 3e^x$ $\quad f'''(0) = 3$

$f^{(4)}(x) = xe^x + 4e^x$ $\quad f^{(4)}(0) = 4$

$$P_4(x) = 0 + x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4$$
$$= x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

20. $f(x) = x^2e^{-x}$ $\qquad f(0) = 0$

$f'(x) = 2xe^{-x} - x^2e^{-x}$ $\qquad f'(0) = 0$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$ $\qquad f''(0) = 2$

$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x}$ $\qquad f'''(0) = -6$

$f^{(4)}(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x}$ $\qquad f^{(4)}(0) = 12$

$$P_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4 = x^2 - x^3 + \frac{1}{2}x^4$$

21. $f(x) = \dfrac{1}{x+1} = (x+1)^{-1}$ $\qquad f(0) = 1$

$f'(x) = -(x+1)^{-2}$ $\qquad f'(0) = -1$

$f''(x) = 2(x+1)^{-3}$ $\qquad f''(0) = 2$

$f'''(x) = -6(x+1)^{-4}$ $\qquad f'''(0) = -6$

$f^{(4)}(x) = 24(x+1)^{-5}$ $\qquad f^{(4)}(0) = 24$

$f^{(5)}(x) = -120(x+1)^{-6}$ $\qquad f^{(5)}(0) = -120$

$$P_5(x) = 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \frac{120x^5}{5!} = 1 - x + x^2 - x^3 + x^4 - x^5$$

22. $f(x) = \dfrac{x}{x+1} = \dfrac{x+1-1}{x+1} = 1 - (x+1)^{-1}$ $\qquad f(0) = 0$

$f'(x) = (x+1)^{-2}$ $\qquad f'(0) = 1$

$f''(x) = -2(x+1)^{-3}$ $\qquad f''(0) = -2$

$f'''(x) = 6(x+1)^{-4}$ $\qquad f'''(0) = 6$

$f^{(4)}(x) = -24(x+1)^{-5}$ $\qquad f^{(4)}(0) = -24$

$$P_4(x) = 0 + 1(x) - \frac{2}{2}x^2 + \frac{6}{6}x^3 - \frac{24}{24}x^4 = x - x^2 + x^3 - x^4$$

23. $f(x) = \sec x$ $\qquad f(0) = 1$

$f'(x) = \sec x \tan x$ $\qquad f'(0) = 0$

$f''(x) = \sec^3 x + \sec x \tan^2 x$ $\qquad f''(0) = 1$

$$P_2(x) = 1 + 0x + \frac{1}{2!}x^2 = 1 + \frac{1}{2}x^2$$

24. $f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = \sec^2 x$ $\qquad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''(0) = 0$

$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $\qquad f'''(0) = 2$

$$P_3(x) = 0 + 1(x) + 0 + \tfrac{2}{6}x^3 = x + \tfrac{1}{3}x^3$$

25. $f(x) = \dfrac{2}{x} = 2x^{-1}$ $\qquad f(1) = 2$

$f'(x) = -2x^{-2}$ $\qquad f'(1) = -2$

$f''(x) = 4x^{-3}$ $\qquad f''(1) = 4$

$f'''(x) = -12x^{-4}$ $\qquad f'''(1) = -12$

$$P_3(x) = 2 - 2(x-1) + \frac{4}{2!}(x-1)^2 - \frac{12}{3!}(x-1)^3 = 2 - 2(x-1) + 2(x-1)^2 - 2(x-1)^3$$

26. $f(x) = \dfrac{1}{x^2} = x^{-2}$ $\qquad f(2) = 1/4$

$f'(x) = -2x^{-3}$ $\qquad f'(2) = -1/4$

$f''(x) = 6x^{-4}$ $\qquad f''(2) = 3/8$

$f'''(x) = -24x^{-5}$ $\qquad f'''(2) = -3/4$

$f^{(4)}(x) = 120x^{-6}$ $\qquad f^{(4)}(2) = 15/8$

$$P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3/8}{2!}(x-2)^2 - \frac{3/4}{3!}(x-2)^3 + \frac{15/8}{4!}(x-2)^4$$
$$= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4$$

27. $f(x) = \sqrt{x} = x^{1/2}$ $\qquad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8}x^{-5/2}$ $\qquad f'''(4) = \frac{3}{256}$

$$P_3(x) = 2 + \frac{1}{4}(x - 4) - \frac{1/32}{2!}(x - 4)^2 + \frac{3/256}{3!}(x - 4)^3$$
$$= 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3$$

28. $f(x) = x^{1/3}$ $\qquad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\qquad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\qquad f''(8) = -\frac{1}{144}$

$f'''(x) = \frac{10}{27}x^{-8/3}$ $\qquad f'''(8) = \frac{10}{27}\cdot\frac{1}{2^8} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 8)^2 + \frac{5}{20{,}736}(x - 8)^3$$

29. $f(x) = \ln x$ $\qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(2) = 1/2$

$f''(x) = -x^{-2}$ $\qquad f''(2) = -1/4$

$f'''(x) = 2x^{-3}$ $\qquad f'''(2) = 1/4$

$f^{(4)}(x) = -6x^{-4}$ $\qquad f^{(4)}(2) = -3/8$

$$P_4(x) = \ln 2 + \frac{1}{2}(x - 2) - \frac{1/4}{2!}(x - 2)^2 + \frac{1/4}{3!}(x - 2)^3 - \frac{3/8}{4!}(x - 2)^4$$
$$= \ln 2 + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \frac{1}{64}(x - 2)^4$$

30. $f(x) = x^2 \cos x$ $\qquad f(\pi) = -\pi^2$

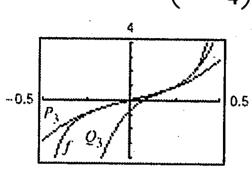
$f'(x) = \cos x - x^2 \sin x$ $\qquad f'(\pi) = -2\pi$

$f''(x) = 2\cos x - 4x \sin x - x^2 \cos x$ $\qquad f''(\pi) = -2 + \pi^2$

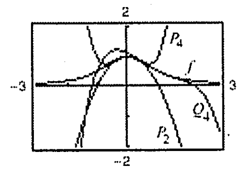
$$P_2(x) = -\pi^2 - 2\pi(x - \pi) + \frac{(\pi^2 - 2)}{2}(x - \pi)^2$$

31. (a) $P_3(x) = \pi x + \frac{\pi^3}{3}x^3$

(b) $Q_3(x) = 1 + 2\pi\left(x - \frac{1}{4}\right) + 2\pi^2\left(x - \frac{1}{4}\right)^2 + \frac{8}{3}\pi^3\left(x - \frac{1}{4}\right)^3$



32. (a) $P_4(x) = 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{24}{4!}x^4 = 1 - x^2 + x^4$



(b) $Q_4(x) = \frac{1}{2} + \left(-\frac{1}{2}\right)(x - 1) + \frac{1/2}{2!}(x - 1)^2 + \frac{0}{3!}(x - 1)^3 + \frac{-3}{4!}(x - 1)^4 = \frac{1}{2} - \frac{1}{2}(x - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{8}(x - 1)^4$

33. $f(x) = \sin x$

$P_1(x) = x$

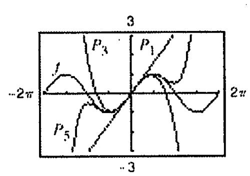
$P_3(x) = x - \frac{1}{6}x^3$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$

(a)

x	0.00	0.25	0.50	0.75	1.00
$\sin x$	0.0000	0.2474	0.4794	0.6816	0.8415
$P_1(x)$	0.0000	0.2500	0.5000	0.7500	1.0000
$P_3(x)$	0.0000	0.2474	0.4792	0.6797	0.8333
$P_5(x)$	0.0000	0.2474	0.4794	0.6817	0.8417

(b)



(c) As the distance increases, the accuracy decreases.

34. (a) $f(x) = e^x \qquad f(1) = e$

$f'(x) = e^x \qquad f'(1) = e$

$f''(x) = f'''(x) = f^{(4)}(x) = e^x$ and $f''(1) = f'''(1) = f^{(4)}(1) = e$

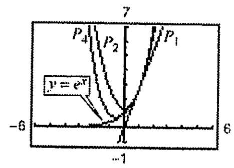
$P_1(x) = e + e(x - 1)$

$P_2(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2$

$P_4(x) = e + e(x - 1) + \frac{e}{2}(x - 1)^2 + \frac{e}{6}(x - 1)^3 + \frac{e}{24}(x - 1)^4$

x	1.00	1.25	1.50	1.75	2.00
e^x	e	3.4903	4.4817	5.7546	7.3891
$P_1(x)$	e	3.3979	4.0774	4.7570	5.4366
$P_2(x)$	e	3.4828	4.4172	5.5215	6.7957
$P_4(x)$	e	3.4903	4.4809	5.7485	7.3620

(b)



(c) As the degree increases, the accuracy increases. As the distance from x to 1 increases, the accuracy decreases.

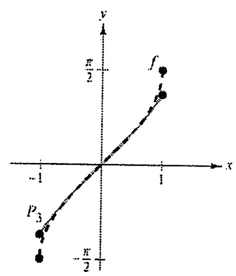
35. $f(x) = \arcsin x$

(a) $P_3(x) = x + \dfrac{x^3}{6}$

(b)

x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$	−0.848	−0.524	−0.253	0	0.253	0.524	0.848
$P_3(x)$	−0.820	−0.521	−0.253	0	0.253	0.521	0.820

(c)



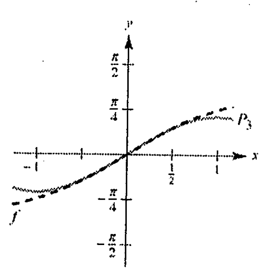
36. (a) $f(x) = \arctan x$

$P_3(x) = x - \dfrac{x^3}{3}$

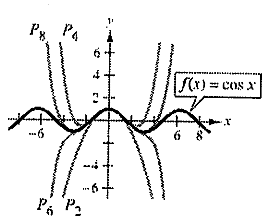
(b)

x	−0.75	−0.50	−0.25	0	0.25	0.50	0.75
$f(x)$	−0.6435	−0.4636	−0.2450	0	0.2450	0.4636	0.6435
$P_3(x)$	−0.6094	−0.4583	−0.2448	0	0.2448	0.4583	0.6094

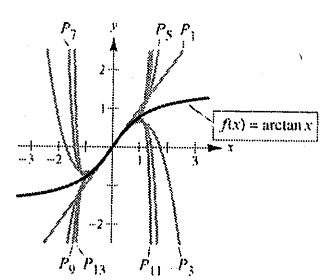
(c)



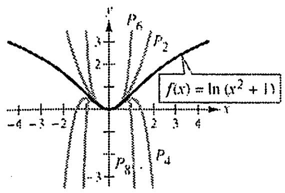
37. $f(x) = \cos x$



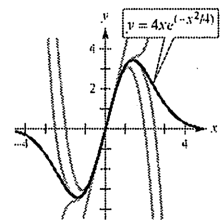
38. $f(x) = \arctan x$



39. $f(x) = \ln(x^2 + 1)$



40. $f(x) = 4xe^{-x^2/4}$



41. $f(x) = e^{3x} \approx 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4$

$f\left(\frac{1}{2}\right) \approx 4.3984$

42. $f(x) = x^2e^{-x} \approx x^2 - x^3 + \frac{1}{2}x^4$

$f\left(\frac{1}{5}\right) \approx 0.0328$

43. $f(x) = \ln x \approx \ln(2) + \frac{1}{2}(x - 2) - \frac{1}{8}(x - 2)^2 + \frac{1}{24}(x - 2)^3 - \frac{1}{64}(x - 2)^4$

$f(2.1) \approx 0.7419$

44. $f(x) = x^2 \cos x \approx -\pi^2 - 2\pi(x - \pi) + \left(\frac{\pi^2 - 2}{2}\right)(x - \pi)^2$

$f\left(\frac{7\pi}{8}\right) \approx -6.7954$

45. $f(x) = \cos x$; $f^{(5)}(x) = -\sin x \Rightarrow$ Max on [0, 0.3] is 1.

$R_4(x) \le \frac{1}{5!}(0.3)^5 = 2.025 \times 10^{-5}$

Note: you could use $R_5(x)$: $f^{(6)}(x) = -\cos x$, max on [0, 0.3] is 1.

$R_5(x) \le \frac{1}{6!}(0.3)^6 = 1.0125 \times 10^{-6}$

Exact error: $0.000001 = 1.0 \times 10^{-6}$

46. $f(x) = e^x$; $f^{(6)}(x) = e^x \Rightarrow$ Max on [0, 1] is e^1.

$R_5(x) \le \frac{e^1}{6!}(1)^6 \approx 0.00378 = 3.78 \times 10^{-3}$

47. $f(x) = \arcsin x$; $f^{(4)}(x) = \dfrac{x(6x^2 + 9)}{(1 - x^2)^{7/2}} \Rightarrow$ Max on

[0, 0.4] is $f^{(4)}(0.4) \approx 7.3340$.

$R_3(x) \le \frac{7.3340}{4!}(0.4)^4 \approx 0.00782 = 7.82 \times 10^{-3}$. The exact error is 8.5×10^{-4}. [Note: You could use R_4.]

48. $f(x) = \arctan x$; $f^{(4)}(x) = \dfrac{24x(x^2 + 1)}{(1 - x^2)^4}$

$\Rightarrow$ Max on [0, 0.4] is $f^{(4)}(0.4) \approx 22.3672$.

$R_3(x) \le \frac{22.3672}{4!}(0.4)^4 \approx 0.0239$

49. $g(x) = \sin x$

$\left|g^{(n+1)}(x)\right| \le 1$ for all x.

$R_n(x) \le \frac{1}{(n + 1)!}(0.3)^{n+1} < 0.001$

By trial and error, $n = 3$.

50. $f(x) = \cos x$

$\left|f^{(n+1)}(x)\right| \le 1$ for all x and all n.

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)x^{n+1}}{(n+1)!}\right| \le \frac{(0.1)^{n+1}}{(n+1)!} < 0.001$$

By trial and error, $n = 2$.

51. $f(x) = e^x$

$f^{(n+1)}(x) = e^x$

Max on $[0, 0.6]$ is $e^{0.6} \approx 1.8221$.

$$R_n \le \frac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$$

By trial and error, $n = 5$.

52. $f(x) = \ln x, f'(x) = x^{-1}, f''(x) = -x^{-2}, \ldots$

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$$

The maximum value of $\left|f^{(n+1)}(x)\right|$ on $[1, 1.25]$ is $n!$

$$\left|R_n\right| \le \frac{n!}{(n+1)!}(0.25)^{n+1} < 0.001$$

$$\frac{(0.25)^{n+1}}{n+1} < 0.001$$

By trial and error, $n = 3$

53. $f(x) = \ln(x+1)$

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(x+1)^{n+1}} \Rightarrow \text{Max on } [0, 0.5] \text{ is } n!.$$

$$R_n \le \frac{n!}{(n+1)!}(0.5)^{n+1} = \frac{(0.5)^{n+1}}{n+1} < 0.0001$$

By trial and error, $n = 9$. (See Example 9.) Using 9 terms, $\ln(1.5) \approx 0.4055$.

54. $f(x) = e^{-\pi x}, f(1.3)$

$f'(x) = (-\pi)e^{-\pi x}$

$$f^{(n+1)}(x) = (-\pi)^{n+1}e^{-\pi x} \le \left|(-\pi)^{n+1}\right| \text{ on } [0, 1.3]$$

$$\left|R_n\right| \le \frac{(\pi)^{n+1}}{(n+1)!}(1.3)^{n+1} < 0.0001$$

By trial and error, $n = 16$. Using 16 terms, $e^{-\pi(1.3)} \approx 0.01684$.

55. $f(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}, x < 0$

$$R_3(x) = \frac{e^z}{4!}x^4 < 0.001$$

$e^z x^4 < 0.024$

$\left|xe^{z/4}\right| < 0.3936$

$$\left|x\right| < \frac{0.3936}{e^{z/4}} < 0.3936, z < 0$$

$-0.3936 < x < 0$

56. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$$\left|R_3(x)\right| = \left|\frac{\sin z}{4!}x^4\right| \le \frac{\left|x^4\right|}{4!} < 0.001$$

$x^4 < 0.024$

$\left|x\right| < 0.3936$

$-0.3936 < x < 0.3936$

57. $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, fifth degree polynomial

$\left|f^{(n+1)}(x)\right| \le 1$ for all x and all n.

$$\left|R_5(x)\right| \le \frac{1}{6!}\left|x\right|^6 < 0.001$$

$\left|x\right|^6 < 0.72$

$\left|x\right| < 0.9467$

$-0.9467 < x < 0.9467$

Note: Use a graphing utility to graph $y = \cos x - \left(1 - x^2/2 + x^4/24\right)$ in the viewing window $[-0.9467, 0.9467] \times [-0.001, 0.001]$ to verify the answer.

58. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

$f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x},$

$f'''(x) = -8e^{-2x}, f^{(4)}(x) = 16e^{-2x}$

$$R_3(x) = \frac{f^4(z)}{4!}(x-0)^4 = \frac{16e^{-2z}}{24}x^4 = \frac{2}{3}e^{-2z}x^4 < 0.001$$

$e^{-2z}x^4 < 0.0015$

$$x < \left(\frac{0.0015}{e^{-2z}}\right)^{1/4} \approx 0.1970e^{2z} < 0.1970, \text{ for } z < 0.$$

So, $0 < x < 0.1970$.

In fact, by graphing $f(x) = e^{-2x}$ and $y = 1 - 2x + 2x^2 - \frac{4}{3}x^3$, you can verify that $\left|f(x) - y\right| < 0.001$ on $(-0.19294, 0.20068)$.

59. The graph of the approximating polynomial P and the elementary function f both pass through the point $(c, f(c))$ and the slopes of P and f agree at $(c, f(c))$. Depending on the degree of P, the nth derivatives of P and f agree at $(c, f(c))$.

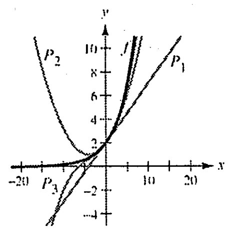
60. $f(c) = P_2(c)$, $f'(c) = P_2'(c)$, and $f''(c) = P_2''(c)$

61. See definition on page 638.

62. See Theorem 9.19, page 642.

63. As the degree of the polynomial increases, the graph of the Taylor polynomial becomes a better and better approximation of the function within the interval of convergence. Therefore, the accuracy is increased.

64.



65. (a) $f(x) = e^x$

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$g(x) = xe^x$$

$$Q_5(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$$

$$Q_5(x) = x\,P_4(x)$$

(b) $f(x) = \sin x$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$g(x) = x \sin x$$

$$Q_6(x) = x\,P_5(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

(c) $$g(x) = \frac{\sin x}{x} = \frac{1}{x}P_5(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$

66. (a) $P_5(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$ for $f(x) = \sin x$

$$P_5'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

This is the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) $Q_6(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!}$ for $\cos x$

$$Q_6'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} = -P_5(x)$$

(c) $$R(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The first four terms are the same!

67. (a) $$Q_2(x) = -1 + \frac{\pi^2(x + 2)^2}{32}$$

(b) $$R_2(x) = -1 + \frac{\pi^2(x - 6)^2}{32}$$

(c) No. The polynomial will be linear. Horizontal translations of the result in part (a) are possible only at $x = -2 + 8n$ (where n is an integer) because the period of f is 8.

68. Let f be an odd function and P_n be the nth Maclaurin polynomial for f. Because f is odd, f' is even:

$$f'(-x) = \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h}$$

$$= \lim_{h \to 0} \frac{-f(x - h) + f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + (-h)) - f(x)}{-h} = f'(x).$$

Similarly, f'' is odd, f''' is even, etc. Therefore, $f, f'', f^{(4)}$, etc. are all odd functions, which implies that $f(0) = f''(0) = \cdots = 0$. So, in the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ all the coefficients of the even power of x are zero.

69. Let f be an even function and P_n be the nth Maclaurin polynomial for f. Because f is even, f' is odd, f'' is even, f''' is odd, etc. All of the odd derivatives of f are odd and so, all of the odd powers of x will have coefficients of zero. P_n will only have terms with even powers of x.