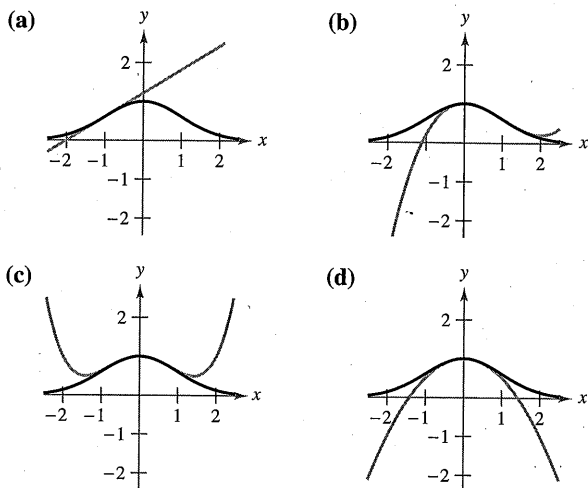


9.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the corresponding graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $g(x) = -\frac{1}{2}x^2 + 1$
2. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$
3. $g(x) = e^{-1/2}[(x+1) + 1]$
4. $g(x) = e^{-1/2}[\frac{1}{3}(x-1)^3 - (x-1) + 1]$

Finding a First-Degree Polynomial Approximation In Exercises 5–8, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1 . What is P_1 called?

5. $f(x) = \frac{\sqrt{x}}{4}, c = 4$
6. $f(x) = \frac{6}{\sqrt[3]{x}}, c = 8$
7. $f(x) = \sec x, c = \frac{\pi}{4}$
8. $f(x) = \tan x, c = \frac{\pi}{4}$

Graphical and Numerical Analysis In Exercises 9 and 10, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2 .

9. $f(x) = \frac{4}{\sqrt{x}}, c = 1$
 $P_2(x) = 4 - 2(x-1) + \frac{3}{2}(x-1)^2$

| | | | | | | | |
|----------|---|-----|-----|---|-----|-----|---|
| x | 0 | 0.8 | 0.9 | 1 | 1.1 | 1.2 | 2 |
| $f(x)$ | | | | | | | |
| $P_2(x)$ | | | | | | | |

10. $f(x) = \sec x, c = \frac{\pi}{4}$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

| | | | | | | | |
|----------|-------|-------|-------|-----------------|-------|-------|-------|
| x | -2.15 | 0.585 | 0.685 | $\frac{\pi}{4}$ | 0.885 | 0.985 | 1.785 |
| $f(x)$ | | | | | | | |
| $P_2(x)$ | | | | | | | |

11. Conjecture Consider the function $f(x) = \cos x$ and its Maclaurin polynomials $P_2, P_4,$ and P_6 (see Example 5).

- (a) Use a graphing utility to graph f and the indicated polynomial approximations.
- (b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 4,$ and 6 .
- (c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

12. Conjecture Consider the function $f(x) = x^2e^x$.

- (a) Find the Maclaurin polynomials $P_2, P_3,$ and P_4 for f .
- (b) Use a graphing utility to graph $f, P_2, P_3,$ and P_4 .
- (c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 3,$ and 4 .
- (d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

Finding a Maclaurin Polynomial In Exercises 13–24, find the n th Maclaurin polynomial for the function.

13. $f(x) = e^{4x}, n = 4$
14. $f(x) = e^{-x}, n = 5$
15. $f(x) = e^{-x/2}, n = 4$
16. $f(x) = e^{x/3}, n = 4$
17. $f(x) = \sin x, n = 5$
18. $f(x) = \cos \pi x, n = 4$
19. $f(x) = xe^x, n = 4$
20. $f(x) = x^2e^{-x}, n = 4$
21. $f(x) = \frac{1}{x+1}, n = 5$
22. $f(x) = \frac{x}{x+1}, n = 4$
23. $f(x) = \sec x, n = 2$
24. $f(x) = \tan x, n = 3$

Finding a Taylor Polynomial In Exercises 25–30, find the n th Taylor polynomial centered at c .

25. $f(x) = \frac{2}{x}, n = 3, c = 1$
26. $f(x) = \frac{1}{x^2}, n = 4, c = 2$
27. $f(x) = \sqrt{x}, n = 3, c = 4$
28. $f(x) = \sqrt[3]{x}, n = 3, c = 8$
29. $f(x) = \ln x, n = 4, c = 2$
30. $f(x) = x^2 \cos x, n = 2, c = \pi$

Finding Taylor Polynomials Using Technology In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function f . Graph the function and the Taylor polynomials.

31. $f(x) = \tan \pi x$ 32. $f(x) = \frac{1}{x^2 + 1}$
- (a) $n = 3, c = 0$ (a) $n = 4, c = 0$
 (b) $n = 3, c = 1/4$ (b) $n = 4, c = 1$

33. Numerical and Graphical Approximations

- (a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

| | | | | | |
|----------|---|--------|--------|--------|--------|
| x | 0 | 0.25 | 0.50 | 0.75 | 1.00 |
| $\sin x$ | 0 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |
| $P_1(x)$ | | | | | |
| $P_3(x)$ | | | | | |
| $P_5(x)$ | | | | | |

- (b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).
 (c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

34. Numerical and Graphical Approximations

- (a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = e^x$ centered at $c = 1$ to complete the table.

| | | | | | |
|----------|------|--------|--------|--------|--------|
| x | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
| e^x | e | 3.4903 | 4.4817 | 5.7546 | 7.3891 |
| $P_1(x)$ | | | | | |
| $P_2(x)$ | | | | | |
| $P_4(x)$ | | | | | |

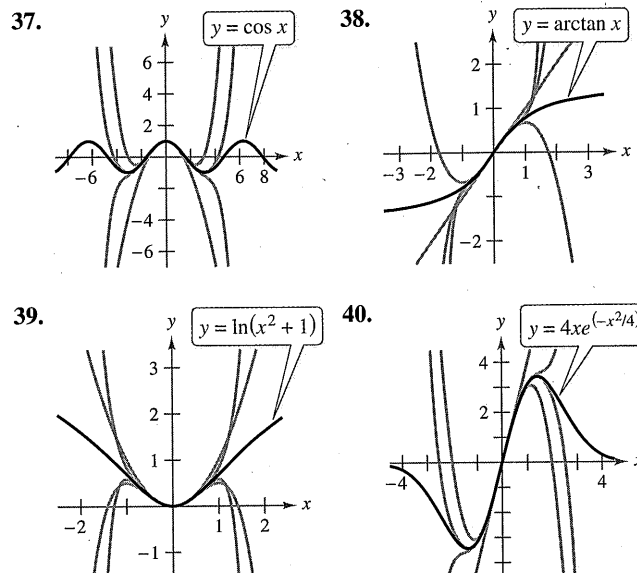
- (b) Use a graphing utility to graph $f(x) = e^x$ and the Taylor polynomials in part (a).
 (c) Describe the change in accuracy of polynomial approximations as the degree increases.

Numerical and Graphical Approximations In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.

| | | | | | | | |
|----------|-------|-------|-------|---|------|------|------|
| x | -0.75 | -0.50 | -0.25 | 0 | 0.25 | 0.50 | 0.75 |
| $f(x)$ | | | | | | | |
| $P_3(x)$ | | | | | | | |

35. $f(x) = \arcsin x$ 36. $f(x) = \arctan x$

Identifying Maclaurin Polynomials In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.



Approximating a Function Value In Exercises 41–44, approximate the function at the given value of x , using the polynomial found in the indicated exercise.

41. $f(x) = e^{4x}$, $f(\frac{1}{4})$, Exercise 13
 42. $f(x) = x^2e^{-x}$, $f(\frac{1}{5})$, Exercise 20
 43. $f(x) = \ln x$, $f(2.1)$, Exercise 29
 44. $f(x) = x^2 \cos x$, $f(\frac{7\pi}{8})$, Exercise 30

Using Taylor's Theorem In Exercises 45–48, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.

45. $\cos(0.3) \approx 1 - \frac{(0.3)^2}{2!} + \frac{(0.3)^4}{4!}$
 46. $e \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}$
 47. $\arcsin(0.4) \approx 0.4 + \frac{(0.4)^3}{2 \cdot 3}$
 48. $\arctan(0.4) \approx 0.4 - \frac{(0.4)^3}{3}$

Finding a Degree In Exercises 49–52, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.

49. $\sin(0.3)$
 50. $\cos(0.1)$
 51. $e^{0.6}$
 52. $\ln(1.25)$

Finding a Degree Using Technology In Exercises 53 and 54, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.0001. Use a computer algebra system to obtain and evaluate the required derivative.

53. $f(x) = \ln(x + 1)$, approximate $f(0.5)$.

54. $f(x) = e^{-\pi x}$, approximate $f(1.3)$.

Finding Values In Exercises 55–58, determine the values of x for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.

55. $f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $x < 0$

56. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

57. $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

58. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

WRITING ABOUT CONCEPTS

59. Polynomial Approximation An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c* .

60. Polynomial Approximation When an elementary function f is approximated by a second-degree polynomial P_2 centered at c , what is known about f and P_2 at c ? Explain your reasoning.

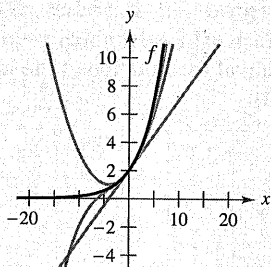
61. Taylor Polynomial State the definition of an n th-degree Taylor polynomial of f centered at c .

62. Accuracy of a Taylor Polynomial Describe the accuracy of the n th-degree Taylor polynomial of f centered at c as the distance between c and x increases.

63. Accuracy of a Taylor Polynomial In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial increases? Explain your reasoning.

64.

HOW DO YOU SEE IT? The graphs show first-, second-, and third-degree polynomial approximations P_1 , P_2 , and P_3 of a function f . Label the graphs of P_1 , P_2 , and P_3 . To print an enlarged copy of the graph, go to MathGraphs.com.



65. Comparing Maclaurin Polynomials

(a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions $f(x) = e^x$ and $g(x) = xe^x$. What is the relationship between them?

(b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.

(c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.

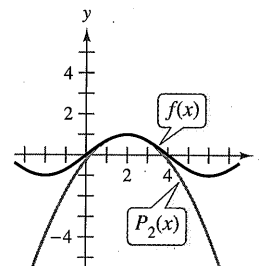
66. Differentiating Maclaurin Polynomials

(a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.

(c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

67. Graphical Reasoning The figure shows the graphs of the function $f(x) = \sin(\pi x/4)$ and the second-degree Taylor polynomial $P_2(x) = 1 - (\pi^2/32)(x - 2)^2$ centered at $x = 2$.



(a) Use the symmetry of the graph of f to write the second-degree Taylor polynomial $Q_2(x)$ for f centered at $x = -2$.

(b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial $R_2(x)$ for f centered at $x = 6$.

(c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for f centered at $x = 4$? Explain.

68. Proof Prove that if f is an odd function, then its n th Maclaurin polynomial contains only terms with odd powers of x .

69. Proof Prove that if f is an even function, then its n th Maclaurin polynomial contains only terms with even powers of x .

70. Proof Let $P_n(x)$ be the n th Taylor polynomial for f at c . Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \leq k \leq n$. (See Exercises 9 and 10.)

71. Writing The proof in Exercise 70 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.