① $f(x)=e^{2x},\ f(3)=e^6 \longrightarrow = 2^0e^6$
$f'(x)=2e^{2x},\ f'(3)=2e^6 \longrightarrow = 2^1e^6$
$f''(x)=4e^{2x},\ f''(3)=4e^6 \longrightarrow = 2^2e^6$
$f'''(x)=8e^{2x},\ f'''(3)=8e^6 \longrightarrow = 2^3e^6$
$f^{(4)}(x)=16e^{2x},\ f^{(4)}(3)=16e^6 \longrightarrow = 2^4e^6$

*sometimes it's easier to find the pattern for the n^{th} term before you simplify each term

$$e^{2x}= e^6+2e^6(x-3)+\frac{4e^6}{2!}(x-3)^2+\frac{8e^6}{3!}(x-3)^3+\cdots$$
$$= e^6+2^1e^6(x-3)+\frac{2^2e^6}{2!}(x-3)^2+\frac{2^3e^6}{3!}(x-3)^3+\frac{2^4e^6}{4!}(x-3)^3+\cdots+\frac{2^ne^6}{n!}(x-3)^n+\cdots$$

② $f(x)=\frac{1}{x},\ f(1)=1 \longrightarrow = 0! = (-1)^0 0!$
$f'(x)=-\frac{1}{x^2},\ f'(1)=-1 \Longrightarrow = -1! = (-1)^1 1!$
$f''(x)=\frac{2}{x^3},\ f''(1)=2 \longrightarrow = 2! = (-1)^2 2!$
$f'''(x)=\frac{-6}{x^4},\ f'''(1)=-6 \longrightarrow = -3! = (-1)^3 3!$
$f^{(4)}(x)=\frac{24}{x^5},\ f^{(4)}(1)=24 \longrightarrow = 4! = (-1)^4 4!$

$$\frac{1}{x}=1-1(x-1)+\frac{2}{2!}(x-1)^2-\frac{6}{3!}(x-1)^3+\frac{24}{4!}(x-1)^4+\cdots+\frac{(-1)^n n!}{n!}(x-1)^n+\cdots$$
$$=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4+\cdots+(-1)^n(x-1)^n+\cdots$$

③ $f(x)=\ln x,\ f(1)=0$
$f'(x)=\frac{1}{x},\ f'(1)=1$
$f''(x)=-\frac{1}{x^2},\ f''(1)=-1$
$f'''(x)=\frac{2}{x^3},\ f'''(1)=2$
$f^{(4)}(x)=-\frac{6}{x^4},\ f^{(4)}(1)=-6$

*sometimes it's easier to find the pattern for the n^{th} term after you simplify each term.

$$\ln x = 0+1(x-1)-\frac{1}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3-\frac{6}{4!}(x-1)^4+\cdots$$
$$=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots+\frac{(-1)^{n+1}}{n}(x-1)^n+\cdots$$

(4) $f(x) = \sin x,\ a = \frac{\pi}{6}$

$f(x) = \sin x,\ f\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos x,\ f'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x,\ f''\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos x,\ f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$f^{(4)}(x) = \sin x,\ f^{(4)}\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1/2}{2!}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}/2}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{1/2}{4!}\left(x - \frac{\pi}{6}\right)^4 + \cdots$$

(5) $f(x) = \cos x,\ a = -\frac{\pi}{4}$

$f(x) = \cos x,\ f\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x,\ f'\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''(x) = -\cos x,\ f''\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin x,\ f'''\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \cos x,\ f^{(4)}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$$\cos x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x + \frac{\pi}{4}\right) - \frac{\sqrt{2}/2}{2!}\left(x + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}/2}{3!}\left(x + \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}/2}{4!}\left(x + \frac{\pi}{4}\right)^4 + \cdots$$

(6) $f(x) = e^{-\frac{x}{2}}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

this n starts at $n=0$.

$$e^{-\frac{x}{2}} = 1 + \left(-\frac{x}{2}\right) + \frac{1}{2!}\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x}{2}\right)^3 + \frac{1}{4!}\left(-\frac{x}{2}\right)^4 + \cdots + \frac{1}{n!}\left(-\frac{x}{2}\right)^n + \cdots$$

$$e^{-\frac{x}{2}} = 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \cdots + \frac{(-1)^n x^n}{2^n \cdot n!} + \cdots$$

* it is easy to adjust an existing n^{th} term when creating new series to get the new n^{th} term

(7) $f(x) = \sin(x^2)$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!} + \cdots$$

this n starts at $n=1$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^{n+1} x^{4n-2}}{(2n-1)!} + \cdots$$

(8) $f(x) = \frac{\cos(3x)}{x}$, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1}(x^{2n-2})}{(2n-2)!} + \cdots$

$$\cos(3x) = 1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \frac{3^6 x^6}{6!} + \cdots + \frac{(-1)^{n+1} \cdot 3^{(2n-2)} x^{2n-2}}{(2n-2)!} + \cdots$$

$$\frac{\cos(3x)}{x} = \frac{1}{x} - \frac{3^2 x}{2!} + \frac{3^4 x^3}{4!} - \frac{3^6 x^5}{6!} + \cdots + \frac{(-1)^{n+1} 3^{2n-2} x^{2n-3}}{(2n-2)!} + \cdots$$

(9) $f(x) = x^2 e^{-x}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots$$ ← n starts at one here

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^{n+1} x^{n-1}}{(n-1)!} + \cdots$$

$$x^2 e^{-x} = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \cdots + \frac{(-1)^{n+1} x^{n+1}}{(n-1)!} + \cdots$$

* for the n^{th} term, the "n" can reference the first non-zero term in the series. this first term can correspond to either $n=0$ or $n=1$ (typically) as long as the n^{th} term is consistent

(10) $f(x) = \sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^{2n}(-1)^n}{(2n)!} + \cdots$$ ← n starts at zero here

$$\cos 2x = 1 - \frac{2^2 x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \cdots + \frac{2^{2n} x^{2n} (-1)^n}{(2n)!}$$

$$\frac{1}{2}\cos 2x = \frac{1}{2} - \frac{2x^2}{2!} + \frac{2^3 x^4}{4!} - \frac{2^5 x^6}{6!} + \cdots + \frac{2^{2n-1} x^{2n} (-1)^n}{(2n)!}$$

$$\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} + \cdots + \frac{2^{2n-1} x^{2n} (-1)^{n+1}}{(2n)!} + \cdots$$

$$\sin^2 x = \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \frac{2^7 x^8}{8!} + \cdots + \frac{2^{2n-1} x^{2n} (-1)^{n+1}}{(2n)!}$$ ← n starts at one here

(11) $$\int_0^1 \sin(x^2)\,dx \approx \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^{n+1} x^{4n-2}}{(2n-1)!} + \cdots\right) dx$$

$$= \frac{1}{3}x^3 - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots + \frac{(-1)^{n+1} x^{4n-1}}{(4n-1)(2n-1)!} + \cdots \Big|_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \cdots + \frac{(-1)^{n+1}}{(4n-1)(2n-1)!}\right) - (0)$$

$$\approx \boxed{0.3102}$$

(12) $f(x) = \sqrt{1+x}$, $c = 0$

(a) $f(x) = (1+x)^{1/2}$, $f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$, $f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, $f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$, $f'''(0) = \frac{3}{8}$

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$, $f^{(4)}(0) = -\frac{15}{16}$

$$\boxed{f(x) = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1/4}{2!}x^2 + \frac{3/8}{3!}x^3 - \frac{15/16}{4!}x^4 + \cdots}$$

(b) $g(x) = \sqrt{1+x^3} = 1 + \frac{1}{2}(x^3) - \frac{1/4}{2!}(x^3)^2 + \frac{3/8}{3!}(x^3)^3 - \frac{15/16}{4!}(x^3)^4 + \cdots$

$$\boxed{\sqrt{1+x^3} = 1 + \frac{1}{2}x^3 - \frac{1}{4 \cdot 2!}x^6 + \frac{3}{8 \cdot 3!}x^9 - \frac{15}{16 \cdot 4!}x^{12} + \cdots}$$

(c) $h(x) = \int h'(x)\,dx = \int \sqrt{1+x^3}\,dx = \int \left(1 + \frac{1}{2}x^3 - \frac{1}{4 \cdot 2!}x^6 + \frac{3}{8 \cdot 3!}x^9 - \frac{15}{16 \cdot 4!}x^{12} + \cdots\right)dx$

$= C + x + \frac{1}{4 \cdot 2}x^4 - \frac{1}{7 \cdot 4 \cdot 2!}x^7 + \frac{3}{10 \cdot 8 \cdot 3!}x^{10} - \frac{15}{13 \cdot 16 \cdot 4!}x^{13} + \cdots$ *put +C in front (more stylish)

for $h(0) = 4$: $4 = C + 0 + 0 - 0 + \cdots$, $\boxed{C = 4}$

$$\boxed{\text{so } h(x) = 4 + x + \frac{1}{8}x^4 - \frac{1}{56}x^7 + \frac{1}{160}x^{10} - \frac{5}{1664}x^{13} + \cdots}$$

(13) $f(x) = \frac{1}{x-1}$, $c = 2$

(a) $f(x) = (x-1)^{-1}$, $f(2) = 1$

$f'(x) = -(x-1)^{-2}$, $f'(2) = -1$

$f''(x) = 2(x-1)^{-3}$, $f''(2) = 2$

$f'''(x) = -6(x-1)^{-4}$, $f'''(2) = -6$

$f^{(4)}(x) = 24(x-1)^{-5}$, $f^{(4)}(2) = 24$

so $f(x) = 1 - (x-2) + \frac{2}{2!}(x-2)^2 - \frac{6}{3!}(x-2)^3 + \frac{24}{4!}(x-2)^4 + \cdots + \frac{(-1)^n n!}{n!}(x-2)^n + \cdots$

$f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + (x-2)^4 + \cdots + (-1)^n(x-2)^n + \cdots$

(b) $\ln|x-1| = \int \frac{1}{x-1}\,dx = \int \left(1 - (x-2) + (x-2)^2 - (x-2)^3 + (x-2)^4 + \cdots + (-1)^n(x-2)^n + \cdots\right)dx$

$= C + x - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \frac{1}{5}(x-2)^5 + \cdots + \frac{(-1)^n}{n+1}(x-2)^{n+1}$

find C: we know the coordinate of $\ln|x-1|$ at $x = 2$ is $\ln 1 = 0$, so

$0 = C + 2 - 0 + 0 - 0 \cdots$, so $\boxed{C = -2}$ and $C + x = -2 + x = (x-2)$

n starts at zero

and $\ln|x-1| = (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \cdots + \frac{(-1)^n}{n+1}(x-2)^{n+1} + \cdots$