70. Let

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n$$

where $a_i = \dfrac{f^{(i)}(c)}{i!}$.

$$P_n(c) = a_0 = f(c)$$

For

$$1 \le k \le n, \quad P_n^{(k)}(c) = a_n k! = \left(\frac{f^{(k)}(c)}{k!}\right)k! = f^{(k)}(c).$$

71. As you move away from $x = c$, the Taylor Polynomial becomes less and less accurate.

Section 9.8 Power Series

1. Centered at 0

2. Centered at 0

3. Centered at 2

4. Centered at π

5. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right||x| = |x|$$

$|x| < 1 \Rightarrow R = 1$

6. $\displaystyle\sum_{n=0}^{\infty} (3x)^n$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}|3x| = 3|x|$$

$3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4x)^{n+1}/(n+1)^2}{(4x)^n/n^2}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}(4x)\right| = 4|x|$$

$4|x| < 1 \Rightarrow R = \frac{1}{4}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}/5^{n+1}}{(-1)^n x^n/5^n}\right| = \lim_{n\to\infty}\frac{|x|}{5} = \frac{|x|}{5}$$

$\frac{|x|}{5} < 1 \Rightarrow R = 5$

9. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{(2n+2)}/(2n+2)!}{x^{2n}/(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$$

So, the series converges for all $x \Rightarrow R = \infty$.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!x^{2n}}{n!}$

$$L = \lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}/(n+1)!}{(2n)!x^{2n}/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)x^2}{(n+1)}\right| = \infty$$

The series only converges at $x = 0$. $R = 0$.

11. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{4}\right)^n$

Because the series is geometric, it converges only if $\left|\frac{x}{4}\right| < 1$, or $-4 < x < 4$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

12. $\sum_{n=0}^{\infty} (2x)^n$

Because the series is geometric, it converges only if $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$.

13. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{nx}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

When $x = -1$, the p-series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Therefore, the interval of convergence is $(-1, 1]$.

14. $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+2)x^{n+1}}{(-1)^n(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x}{n+1}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)$ diverges.

When $x = -1$, the series $\sum_{n=0}^{\infty}-(n+1)$ diverges.

Therefore, the interval of convergence is $(-1, 1)$.

15. $\sum_{n=0}^{\infty}\frac{x^{5n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5(n+1)}/(n+1)!}{5^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^5}{n+1}\right| = 0$$

The series converges for all x. The interval of convergence is $(-\infty, \infty)$.

16. $\sum_{n=0}^{\infty}\frac{(3x)^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(3x)^n}\right| = \lim_{n\to\infty}\left|\frac{3x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

17. $\sum_{n=0}^{\infty}(2n)!\left(\frac{x}{3}\right)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!(x/3)^{n+1}}{(2n)!(x/3)^n}\right| = \left|\frac{(2n+2)(2n+1)x}{3}\right| = \infty$$

The series converges only for $x = 0$.

18. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(n+1)(n+2)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+2)(n+3)}\cdot\frac{(n+1)(n+2)}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x}{n+3}\right| = |x|$$

Interval: $-1 < x < 1$

When $x = 1$, the alternating series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(n+2)}$ converges.

When $x = -1$, the series $\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}$ converges by limit comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Therefore, the interval of convergence is $[-1, 1]$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

Because the series is geometric, it converges only if

$\left|\frac{x}{6}\right| < 1 \Rightarrow |x| < 6$ or $-6 < x < 6$.

20. $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)!(x-5)^{n+1}/3^{n+1}}{(-1)^n n!(x-5)^n/3^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-5)}{3}\right| = \infty$$

The series converges only for $x = 5$.

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-4)^{n+1}/((n+1)9^{n+1})}{(-1)^n(x-4)^n/(n9^n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\frac{(x-4)}{9}\right| = \frac{1}{9}|x-4|$$

$R = 9$

Interval: $-5 < x < 13$

When $x = 13$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}9^n}{n9^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

When $x = -5$, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-9)^n}{n9^n} = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges.

Therefore, the interval of convergence is $(-5, 13]$.

22. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+2}/[(n+2)4^{n+2}]}{(x-3)^{n+1}/[(n+1)4^{n+1}]}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)(n+1)}{4(n+2)}\right| = \left|\frac{x-3}{4}\right|$$

$R = 4$

Interval: $-1 < x < 7$

When $x = 7$, $\sum_{n=0}^{\infty} \frac{4^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-4)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)}$ converges.

Therefore, the interval of convergence is $[-1, 7)$.

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2}\cdot\frac{n+1}{(-1)^{n+1}(x-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)}{n+2}\right| = |x-1|$$

$R = 1$

Center: $x = 1$

Interval: $-1 < x - 1 < 1$ or $0 < x < 2$

When $x = 0$, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges by the integral test.

When $x = 2$, the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges.

Therefore, the interval of convergence is $(0, 2]$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-2)^{n+1}}{(n+1)2^{n+1}} \Big/ \frac{(-1)^{n+1}(x-2)^n}{n2^n}\right| = \lim_{n\to\infty}\left|\frac{x-2}{2}\frac{n}{n+1}\right| = \left|\frac{x-2}{2}\right|$$

$$\left|\frac{x-2}{2}\right| < 1 \Rightarrow -2 < x-2 < 2 \Rightarrow 0 < x < 4$$

when $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)(2^n)}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)}{n} \text{ diverges.}$$

when $x = 4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

Therefore the interval of convergence is $(0, 4]$.

25. $\sum_{n=1}^{\infty}\left(\frac{x-3}{3}\right)^{n-1}$ is geometric. It converges if

$$\left|\frac{x-3}{3}\right| < 1 \Rightarrow |x-3| < 3 \Rightarrow 0 < x < 6.$$

Therefore, the interval of convergence is $(0, 6)$.

26. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)}\cdot\frac{(2n+1)}{(-1)^n x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+1)}{(2n+3)}x^2\right| = |x^2|$$

$R = 1$

Interval: $-1 < x < 1$

When $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

When $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ converges.

Therefore, the interval of convergence is $[-1, 1]$.

27. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2x)^n}{n+2}\cdot\frac{n+1}{n(-2x)^{n-1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2x)(n+1)^2}{n(n+2)}\right| = 2|x|$$

$R = \frac{1}{2}$

Interval: $-\frac{1}{2} < x < \frac{1}{2}$

When $x = -\frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges by the nth Term Test.

When $x = \frac{1}{2}$, the alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1} \text{ diverges.}$$

Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

28. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{n+1}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

29. $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+4}/(3n+4)!}{x^{3n+1}/(3n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^3}{(3n+4)(3n+3)(3n+2)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

30. $\sum_{n=1}^{\infty} \frac{n!x^n}{(2n)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{n!x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)x}{(2n+2)(2n+1)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty, \infty)$.

31. $\sum_{n=1}^{\infty} \frac{2\cdot 3\cdot 4\cdots(n+1)x^n}{n!} = \sum_{n=1}^{\infty}(n+1)x^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}x\right| = |x|$$

Converges if $|x| < 1 \Rightarrow -1 < x < 1$.

At $x = \pm 1$, diverges.

Therefore the interval of convergence is $(-1, 1)$.

32. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)}\left(x^{2n+1}\right)$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2\cdot 4\cdots(2n)(2n+2)x^{2n+3}}{3\cdot 5\cdot 7\cdots(2n+1)(2n+3)}\cdot\frac{3\cdot 5\cdots(2n+1)}{2\cdot 4\cdots(2n)x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)x^2}{(2n+3)}\right| = \left|x^2\right|$$

$R = 1$

When $x = \pm 1$, the series diverges by comparing it to

$$\sum_{n=1}^{\infty}\frac{1}{2n+1}$$

which diverges.

Therefore, the interval of convergence is $(-1, 1)$.

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}{4^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(4n+3)(x-3)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{(-1)^{n+1}\cdot 3\cdot 7\cdot 11\cdots(4n-1)(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(4n+3)(x-3)}{4}\right| = \infty$$

$R = 0$

Center: $x = 3$

Therefore, the series converges only for $x = 3$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x+1)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\Big/\frac{(n)!(x+1)^n}{1\cdot 3\cdot 5\cdots(2n-1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)}{2n+1}\right| = \frac{1}{2}|x+1|$$

Converges if $\frac{1}{2}|x+1| < 1 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1.$

At $x = 1$, $a_n = \dfrac{n!2^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \dfrac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)} > 1$, diverges.

At $x = -3$, $a_n = \dfrac{n!(-2)^n}{1\cdot 3\cdots(2n-1)} = (-1)^n\dfrac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}$, diverges.

Therefore, the interval of convergence is $(-3, 1)$.

35. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(x-c)^n}{c^n}\cdot\frac{c^{n-1}}{(x-c)^{n-1}}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}$ diverges.

When $x = 2c$, the series $\displaystyle\sum_{n=1}^{\infty} 1$ diverges.

Therefore, the interval of convergence is $(0, 2c)$.

36. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$, k is a positive integer.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^k x^{n+1}}{[k(n+1)]!}\Big/\frac{(n!)^k x^n}{(kn)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^k x}{(k+nk)(k-1+nk)\cdots(1+nk)}\right| = \frac{|x|}{k^k}$$

Converges if $\dfrac{|x|}{k^k} < 1 \Rightarrow R = k^k$.

37. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{k}\right)^n$

Because the series is geometric, it converges only if $|x/k| < 1$ or $-k < x < k$.

Therefore, the interval of convergence is $(-k, k)$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

38. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-c)^n}{nc^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-c)^{n+1}}{(n+1)c^{n+1}}\cdot\frac{nc^n}{(-1)^{n+1}(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{n(x-c)}{c(n+1)}\right| = \frac{1}{c}|x-c|$$

$R = c$

Center: $x = c$

Interval: $-c < x - c < c$ or $0 < x < 2c$

When $x = 0$, the p-series $\displaystyle\sum_{n=1}^{\infty}\frac{-1}{n}$ diverges.

When $x = 2c$, the alternating series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges.

Therefore, the interval of convergence is $(0, 2c)$.

39. $\displaystyle\sum_{n=1}^{\infty}\frac{k(k+1)\cdots(k+n-1)x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)\cdots(k+n-1)(k+n)x^{n+1}}{(n+1)!}\cdot\frac{n!}{k(k+1)\cdots(k+n-1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(k+n)x}{n+1}\right| = |x|$$

$R = 1$

When $x = \pm 1$, the series diverges and the interval of convergence is $(-1, 1)$.

$$\left[\frac{k(k+1)\cdots(k+n-1)}{1\cdot 2\cdots n} \ge 1\right]$$

40. $\displaystyle\sum_{n=1}^{\infty}\frac{n!(x-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(x-c)^{n+1}}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}\cdot\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!(x-c)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-c)}{2n+1}\right| = \frac{1}{2}|x-c|$$

$R = 2$

Interval: $-2 < x - c < 2$ or $c - 2 < x < c + 2$

The series diverges at the endpoints. Therefore, the interval of convergence is $(c - 2, c + 2)$.

$$\left[\frac{n!(c+2-c)^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{n!2^2}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)} > 1\right]$$

41. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$

42. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)x^n = \sum_{n=1}^{\infty}(-1)^n(n)x^{n-1}$

43. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

Replace n with $n - 1$.

44. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{2n-1}$

Replace n with $n - 1$.

45. (a) $f(x) = \displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n, (-3, 3)$ (Geometric)

(b) $f'(x) = \displaystyle\sum_{n=1}^{\infty}\frac{n}{3}\left(\frac{x}{3}\right)^{n-1}, (-3, 3)$

(c) $f''(x) = \displaystyle\sum_{n=2}^{\infty}\frac{n(n-1)}{9}\left(\frac{x}{3}\right)^{n-2}, (-3, 3)$

(d) $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{3}{n+1}\left(\frac{x}{3}\right)^{n+1}, [-3, 3)$

$$\left[\sum\frac{3}{n+1}\left(\frac{-3}{3}\right)^{n+1} = \sum\frac{(-1)^{n+1}3}{n+1}, \text{ converges}\right]$$

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

46. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}, (0, 10]$

(b) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}, (0, 10)$

(c) $f''(x) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}, (0, 10)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^{n+1}}{n(n+1)5^n}, [0, 10]$

47. (a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}, (0, 2]$

(b) $f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1}(x-1)^n, (0, 2)$

(c) $f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1}n(x-1)^{n-1}, (0, 2)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+2}}{(n+1)(n+2)}, [0, 2]$

48. (a) $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}, (1, 3]$

(b) $f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}(x-2)^{n-1}, (1, 3)$

(c) $f''(x) = \sum_{n=2}^{\infty} (-1)^{n+1}(n-1)(x-2)^{n-2}, (1, 3)$

(d) $\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^{n+1}}{n(n+1)}, [1, 3]$

49. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a power series centered at c, where c is constant.

50. The set of all values of x for which the power series converges is the interval of convergence. If the power series converges for all x, then the radius of convergence is $R = \infty$. If the power series converges at only c, then $R = 0$. Otherwise, according to Theorem 8.20, there exists a real number $R > 0$ (radius of convergence) such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.

51. The interval of convergence of a power series is the set of all values of x for which the power series converges.

52. A single point, an interval, or the entire real line.

53. You differentiate and integrate the power series term by term. The radius of convergence remains the same. However, the interval of convergence might change.

54. Answers will vary.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges for $-1 \le x < 1$. At $x = -1$, the convergence is conditional because $\sum \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges for $-1 \le x \le 1$. At $x = \pm 1$, the convergence is absolute.

55. Many answers possible.

(a) $\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$ Geometric: $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$ converges for $-1 < x \le 1$

(c) $\sum_{n=1}^{\infty} (2x+1)^n$ Geometric:
$|2x + 1| < 1 \Rightarrow -1 < x < 0$

(d) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n4^n}$ converges for $-2 \le x < 6$

56. (a) $g(1) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

$S_1 = 1, S_2 = \frac{4}{3}, \ldots$

Matches (iii).

(b) $g(2) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \ldots$

$S_1 = 1, S_2 = \frac{5}{3}, \ldots$

Matches (i).

(c) $g(3) = \sum_{n=0}^{\infty} \left(\frac{3}{3}\right)^n = 1 + 1 + 1 + \ldots$

$S_1 = 1, S_2 = 2, \ldots$

Matches (ii).

(d) $g(-2) = \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = 1 - \frac{2}{3} + \frac{4}{9} - \ldots$ (alternating)

$S_1 = 1, S_2 = \frac{1}{3}, S_3 = \frac{7}{9}, \ldots$

Matches (iv).

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

57. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+3)}\right| = 0$$

Therefore, the interval of convergence is $(-\infty\ \infty)$.

$$g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\frac{1}{2n+2} = 0$$

Therefore, the interval of convergence is $(-\infty\ \infty)$.

(b) $f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = g(x)$

(c) $g'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}$

$$= -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = -f(x)$$

(d) $f(x) = \sin x$ and

$g(x) = \cos x$

58. (a) $f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

The series converges for all x. Therefore, the interval of convergence is $(-\infty, \infty)$.

(b) $f'(x) = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = f(x)$

(c) $f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$f(0) = 1$

(d) $f(x) = e^x$

59. $y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}$

$$y' = \sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)x^{2n-1}}{(2n-1)!}$$

$$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = 0$$

60. $y = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!}$

$$y' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

$$y'' = \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}$$

$$y'' + y = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} = 0$$

61. $y = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$

$$y' = \sum_{n=0}^{\infty}\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$y'' = \sum_{n=1}^{\infty}\frac{(2n)x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!} = y$$

$$y'' - y = 0$$

62. $y = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!}$

$$y' = \sum_{n=1}^{\infty}\frac{(2n-2)x^{2n-1}}{(2n-2)!} = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}$$

$$y'' = \sum_{n=1}^{\infty}\frac{(2n-1)x^{2n-2}}{(2n-1)!} = \sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!} = y$$

$$y'' - y = 0$$

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

63. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$ $\quad y' = \sum_{n=1}^{\infty} \frac{2nx^{2n-1}}{2^n n!}$ $\quad y'' = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!}$

$$y'' - xy' - y = \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=1}^{\infty} \frac{2nx^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$$

$$= \sum_{n=1}^{\infty} \frac{2n(2n-1)x^{2n-2}}{2^n n!} - \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{2^n n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(2n+2)(2n+1)x^{2n}}{2^{n+1}(n+1)!} - \frac{(2n+1)x^{2n}}{2^n n!} \cdot \frac{2(n+1)}{2(n+1)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{2(n+1)x^{2n}\left[(2n+1) - (2n+1)\right]}{2^{n+1}(n+1)!} = 0$$

64.

$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n 4nx^{4n-1}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$$

$$y'' = \sum_{n=1}^{\infty} \frac{(-1)^n 4n(4n-1)x^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)}$$

$$y'' + x^2 y = -x^2 + \sum_{n=2}^{\infty} \frac{(-1)^n 4nx^{4n-2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-5)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} + x^2$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4(n+1)x^{4n+2}}{2^{2n+2}(n+1)! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+2}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)} \frac{2^2(n+1)}{2^2(n+1)} = 0$$

65. $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}$

(a) $$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{2k+2}}{2^{2k+2}\left[(k+1)!\right]^2} \cdot \frac{2^{2k}(k!)^2}{(-1)^k x^{2k}}\right| = \lim_{k\to\infty}\left|\frac{(-1)x^2}{2^2(k+1)^2}\right| = 0$$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b)

$$J_0 = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k (k!)^2}$$

$$J_0' = \sum_{k=1}^{\infty} (-1)^k \frac{2kx^{2k-1}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)x^{2k+1}}{4^{k+1}\left[(k+1)!\right]^2}$$

$$J_0'' = \sum_{k=1}^{\infty} (-1)^k \frac{2k(2k-1)x^{2k-2}}{4^k (k!)^2} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(2k+2)(2k+1)x^{2k}}{4^{k+1}\left[(k+1)!\right]^2}$$

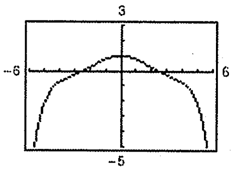
$$x^2 J_0'' + xJ_0' + x^2 J_0 = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2(2k+1)x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{2x^{2k+2}}{4^{k+1}(k+1)!k!} + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+2}}{4^k (k!)^2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2}\left[(-1)\frac{2(2k+1)}{4(k+1)} + (-1)\frac{2}{4(k+1)} + 1\right]$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{4^k (k!)^2}\left[\frac{-4k-2}{4k+4} - \frac{2}{4k+4} + \frac{4k+4}{4k+4}\right] = 0$$

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

(c) $P_6(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$



(d) $\int_0^1 J_0 dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{4^k (k!)^2} dx = \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^k (k!)^2 (2k+1)} \right]_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (k!)^2 (2k+1)} = 1 - \frac{1}{12} + \frac{1}{320} \approx 0.92$

(integral is approximately 0.9197304101)

66. $J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!}$

(a) $\lim_{k \to \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+3}(k+1)!(k+2)!} \cdot \frac{2^{2k+1} k!(k+1)!}{(-1)^k x^{2k+1}} \right| = \lim_{k \to \infty} \left| \frac{(-1)x^2}{2^2(k+2)(k+1)} \right| = 0$

Therefore, the interval of convergence is $-\infty < x < \infty$.

(b) $J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1!)}$

$J_1'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k}}{2^{2k+1} k!(k+1)!}$

$J_1''(x) = \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k-1}}{2^{2k+1} k!(k+1)!}$

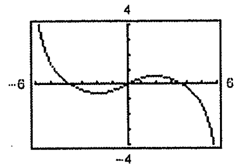
$$\begin{aligned}
x^2 J_1'' + x J_1' + (x^2 - 1) J_1 &= \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \\
&= \left[\sum_{k=1}^{\infty} \frac{(-1)^k (2k+1)(2k) x^{2k+1}}{2^{2k+1} k!(k+1)!} + \frac{x}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k (2k+1) x^{2k+1}}{2^{2k+1} k!(k+1)!} - \frac{x}{2} - \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} \right] \\
&\quad + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} \left[(2k+1)(2k) + (2k+1) - 1 \right]}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1} 4k(k+1)}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k-1}(k-1)!k!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+3}}{2^{2k+1} k!(k+1)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+3}}{2^{2k+1} k!(k+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k+3} \left[(-1) + 1 \right]}{2^{2k+1} k!(k+1)!} = 0
\end{aligned}$$

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

(c) $P_7(x) = \frac{x}{2} - \frac{1}{16}x^3 + \frac{1}{384}x^5 - \frac{1}{18{,}432}x^7$

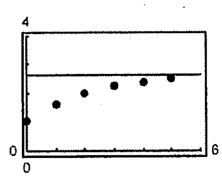
(d) $J_0'(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} 2(k+1) x^{2k+1}}{2^{2k+2}(k+1)!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!}$

$-J_1(x) = -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!(k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{2^{2k+1} k!(k+1)!}$ **Note:** $J_0'(x) = -J_1(x)$

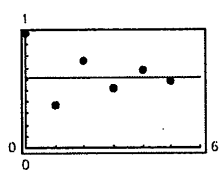


67. $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$, $(-4, 4)$

(a) $\sum_{n=0}^{\infty} \left(\frac{(5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(\frac{5}{8}\right)^n = \frac{1}{1 - 5/8} = \frac{8}{3}$



(b) $\sum_{n=0}^{\infty} \left(\frac{(-5/2)}{4}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{5}{8}\right)^n = \frac{1}{1 + 5/8} = \frac{8}{13}$



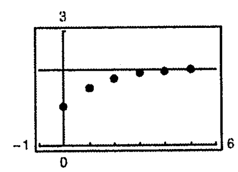
(c) The alternating series converges more rapidly. The partial sums of the series of positive terms approaches the sum from below. The partial sums of the alternating series alternate sides of the horizontal line representing the sum.

(d)

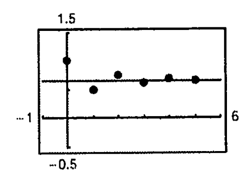
M	10	100	1000	10,000
N	5	14	24	35

68. $\sum_{n=0}^{\infty} (3x)^n$ converges on $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) $x = \frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - (1/2)} = 2$



(b) $x = -\frac{1}{6}: \sum_{n=0}^{\infty} \left(3\left(-\frac{1}{6}\right)\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1 + (1/2)} = \frac{2}{3}$



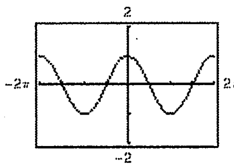
(c) The alternating series converges more rapidly. The partial sums in (a) approach the sum 2 from below. The partial sums in (b) alternate sides of the horizontal line $y = \frac{2}{3}$.

(d) $\sum_{n=0}^{N} \left(3 \cdot \frac{2}{3}\right)^n = \sum_{n=0}^{N} 2^n > M$

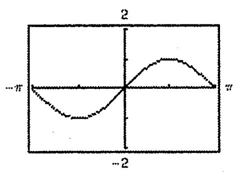
M	10	100	1000	10,000
N	3	6	9	13

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.

69. $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = \cos x$

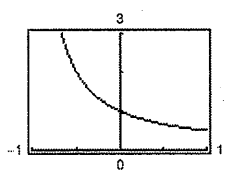


70. $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = \sin x$

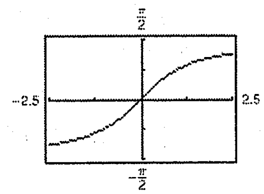


71. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n$ Geometric

$= \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$ for $-1 < x < 1$



72. $f(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = \arctan x, -1 \le x \le 1$



73. False;

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$$

converges for $x = 2$ but diverges for $x = -2$.

74. False; it is not possible. See Theorem 9.20.

75. True; the radius of convergence is $R = 1$ for both series.

76. True

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$$

77. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(n+1+p)!}{(n+1)!(n+1+q)!} x^{n+1} \Big/ \dfrac{(n+p)!}{n!(n+q)!} x^n\right| = \lim_{n\to\infty} \left|\dfrac{(n+1+p)x}{(n+1)(n+1+q)}\right| = 0$

So, the series converges for all x: $R = \infty$.

78. (a) $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$

$S_{2n} = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots + x^{2n} + 2x^{2n+1} = (1 + x^2 + x^4 + \cdots + x^{2n}) + 2x(1 + x^2 + x^4 + \cdots + x^{2n})$

$\lim_{n\to\infty} S_{2n} = \sum_{n=0}^{\infty} x^{2n} + 2x\sum_{n=0}^{\infty} x^{2n}$

Each series is geometric, $R = 1$, and the interval of convergence is $(-1, 1)$.

(b) For $|x| < 1$, $g(x) = \dfrac{1}{1-x^2} + 2x\dfrac{1}{1-x^2} = \dfrac{1+2x}{1-x^2}$.

79. (a) $f(x) = \sum_{n=0}^{\infty} c_n x^n, c_{n+3} = c_n$

$= c_0 + c_1 x + c_2 x^2 + c_0 x^3 + c_1 x^4 + c_2 x^5 + c_0 x^6 + \cdots$

$S_{3n} = c_0(1 + x^3 + \cdots + x^{3n}) + c_1 x(1 + x^3 + \cdots + x^{3n}) + c_2 x^2(1 + x^3 + \cdots + x^{3n})$

$\lim_{n\to\infty} S_{3n} = c_0 \sum_{n=0}^{\infty} x^{3n} + c_1 x \sum_{n=0}^{\infty} x^{3n} + c_2 x^2 \sum_{n=0}^{\infty} x^{3n}$

Each series is geometric, $R = 1$, and the interval of convergence is $(-1, 1)$.

(b) For $|x| < 1$, $f(x) = c_0\dfrac{1}{1-x^3} + c_1 x\dfrac{1}{1-x^3} + c_2 x^2\dfrac{1}{1-x^3} = \dfrac{c_0 + c_1 x + c_2 x^2}{1-x^3}$.

© 2014 Cengage Learning. All Rights Reserved. May not be scanned, copied or duplicated, or posted to a publicly accessible website, in whole or in part.