9.8 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Center of a Power Series In Exercises 1–4, state where the power series is centered.

1. $\sum_{n=0}^{\infty} nx^n$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{2^n n!} x^n$

3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n}}{(2n)!}$

Finding the Radius of Convergence In Exercises 5–10, find the radius of convergence of the power series.

5. $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

6. $\sum_{n=0}^{\infty} (3x)^n$

7. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{5^n}$

9. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

10. $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$

Finding the Interval of Convergence In Exercises 11–34, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

11. $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

12. $\sum_{n=0}^{\infty} (2x)^n$

13. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

14. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

15. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

16. $\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

17. $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$

18. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{6^n}$

20. $\sum_{n=0}^{\infty} \frac{(-1)^n n!(x-5)^n}{3^n}$

21. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-4)^n}{n9^n}$

22. $\sum_{n=0}^{\infty} \frac{(x-3)^{n+1}}{(n+1)4^{n+1}}$

23. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

25. $\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

26. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

27. $\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

28. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

29. $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}$

30. $\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$

31. $\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!}$

32. $\sum_{n=1}^{\infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right] x^{2n+1}$

33. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$

34. $\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Finding the Radius of Convergence In Exercises 35 and 36, find the radius of convergence of the power series, where $c > 0$ and k is a positive integer.

35. $\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

36. $\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$

Finding the Interval of Convergence In Exercises 37–40, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

37. $\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n, \quad k > 0$

38. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

39. $\sum_{n=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)x^n}{n!}, \quad k \geq 1$

40. $\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

Writing an Equivalent Series In Exercises 41–44, write an equivalent series with the index of summation beginning at $n = 1$.

41. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

42. $\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

43. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

44. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

Finding Intervals of Convergence In Exercises 45–48, find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$. Include a check for convergence at the endpoints of the interval.

45. $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

46. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$

47. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

48. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$

WRITING ABOUT CONCEPTS

49. Power Series Define a power series centered at c.

50. Radius of Convergence Describe the radius of convergence of a power series.

51. Interval of Convergence Describe the interval of convergence of a power series.

52. Domain of a Power Series Describe the three basic forms of the domain of a power series.

53. Using a Power Series Describe how to differentiate and integrate a power series with a radius of convergence R. Will the series resulting from the operations of differentiation and integration have a different radius of convergence? Explain.

54. Conditional or Absolute Convergence Give examples that show that the convergence of a power series at an endpoint of its interval of convergence may be either conditional or absolute. Explain your reasoning.

55. Writing a Power Series Write a power series that has the indicated interval of convergence. Explain your reasoning.

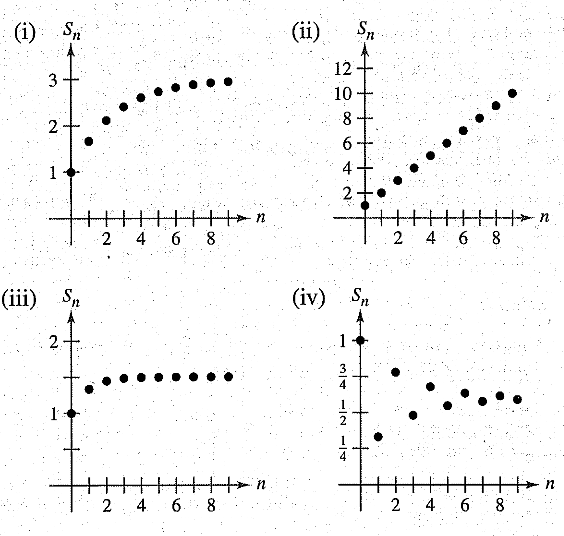
(a) $(-2, 2)$ (b) $(-1, 1]$

(c) $(-1, 0)$ (d) $[-2, 6)$

56. HOW DO YOU SEE IT? Match the graph of the first 10 terms of the sequence of partial sums of the series

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

with the indicated value of the function. [The graphs are labeled (i), (ii), (iii), and (iv).] Explain how you made your choice.



(a) $g(1)$ (b) $g(2)$

(c) $g(3)$ (d) $g(-2)$

57. Using Power Series Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

(a) Find the intervals of convergence of f and g.

(b) Show that $f'(x) = g(x)$.

(c) Show that $g'(x) = -f(x)$.

(d) Identify the functions f and g.

58. Using a Power Series Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(a) Find the interval of convergence of f.

(b) Show that $f'(x) = f(x)$.

(c) Show that $f(0) = 1$.

(d) Identify the function f.

Differential Equation In Exercises 59–64, show that the function represented by the power series is a solution of the differential equation.

59. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad y'' + y = 0$

60. $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad y'' + y = 0$

61. $y = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad y'' - y = 0$

62. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad y'' - y = 0$

63. $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}, \quad y'' - xy' - y = 0$

64. $y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n} n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)},$

$y'' + x^2 y = 0$

65. Bessel Function The Bessel function of order 0 is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k!)^2}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_0'' + x J_0' + x^2 J_0 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_0.

(d) Approximate $\int_0^1 J_0\, dx$ accurate to two decimal places.

66. Bessel Function The Bessel function of order 1 is

$$J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!}.$$

(a) Show that the series converges for all x.

(b) Show that the series is a solution of the differential equation $x^2 J_1'' + x J_1' + (x^2 - 1) J_1 = 0$.

(c) Use a graphing utility to graph the polynomial composed of the first four terms of J_1.

(d) Show that $J_0'(x) = -J_1(x)$.

67. Investigation The interval of convergence of the geometric series $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$ is $(-4, 4)$.

(a) Find the sum of the series when $x = \frac{5}{2}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{5}{2}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N} \left(\frac{5}{4}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

68. Investigation The interval of convergence of the series $\sum_{n=0}^{\infty} (3x)^n$ is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) Find the sum of the series when $x = \frac{1}{6}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{1}{6}$.

(c) Write a short paragraph comparing the rates of convergence of the partial sums with the sums of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number M, there exists a positive integer N such that the partial sum

$$\sum_{n=0}^{N} \left(3 \cdot \frac{2}{3}\right)^n > M.$$

Use a graphing utility to complete the table.

M	10	100	1000	10,000
N				

Identifying a Function In Exercises 69–72, the series represents a well-known function. Use a computer algebra system to graph the partial sum S_{10} and identify the function from the graph.

69. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

70. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

71. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n, \quad -1 < x < 1$

72. $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$

True or False? In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

73. If the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = 2$, then it also converges for $x = -2$.

74. It is possible to find a power series whose interval of convergence is $[0, \infty)$.

75. If the interval of convergence for $\sum_{n=0}^{\infty} a_n x^n$ is $(-1, 1)$, then the interval of convergence for $\sum_{n=0}^{\infty} a_n (x-1)^n$ is $(0, 2)$.

76. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 2$, then

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

77. Proof Prove that the power series

$$\sum_{n=0}^{\infty} \frac{(n+p)!}{n!(n+q)!} x^n$$

has a radius of convergence of $R = \infty$ when p and q are positive integers.

78. Using a Power Series Let

$$g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

where the coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $g(x)$.

79. Using a Power Series Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+3} = c_n$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $f(x)$.

80. Proof Prove that if the power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence of R, then $\sum_{n=0}^{\infty} c_n x^{2n}$ has a radius of convergence of $\sqrt{R}$.

81. Proof For $n > 0$, let $R > 0$ and $c_n > 0$. Prove that if the interval of convergence of the series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

is $[x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 - R$.