① $f(x) = \frac{1}{1+x}$, $a = 0$

Method 1: (Taylor's Rule)

$f(x) = (1+x)^{-1}$, $f(0) = 1$

$f'(x) = -1(1+x)^{-2}$, $f'(0) = -1$

$f''(x) = 2(1+x)^{-3}$, $f''(0) = 2$

$f'''(x) = -6(1+x)^{-4}$, $f'''(0) = -6$

$f^{(4)}(x) = 24(1+x)^{-5}$, $f^{(4)}(0) = 24$

So $\frac{1}{1+x} = 1 - x + \frac{2}{2!}x^2 - \frac{6}{3!}x^3 + \frac{24}{4!}x^4 + \cdots$

$= 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n x^n + \cdots$ or $(-x)^n$

Method 2: (Long division) (for center at zero)

$$\begin{array}{r|l} & 1 - x + x^2 - x^3 + \cdots \\ 1+x & 1 \\ & -(1+x) \\ & \quad -x \\ & -(-x - x^2) \\ & \quad x^2 \\ & -(x^2 + x^3) \\ & \quad -x^3 \end{array}$$

Method 3: (Recognized Geom series)

$S = \frac{a_1}{1-r} = \frac{1}{1+x} = \frac{1}{1-(-x)}$

so first term is 1 and ratio is $-x$

so $f(x) = \sum_{n=0}^{\infty} (-x)^n$

$= 1 - x + x^2 - x^3 + \cdots + (-x)^n + \cdots$

② $f(x) = \frac{1}{1+x^2}$, $a = 0$ (long division)

$$\begin{array}{r|l} & 1 - x^2 + x^4 - x^6 + \cdots \\ 1+x^2 & 1 \\ & -(1+x^2) \\ & \quad -x^2 \\ & -(-x^2 - x^4) \\ & \quad x^4 \end{array}$$

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$

③ $f(x) = \frac{3}{x+2} = \frac{3}{2+x}\left(\frac{1/2}{1/2}\right) = \frac{3/2}{1+\frac{x}{2}} = \frac{3}{2}\left[\frac{1}{1-\left(-\frac{x}{2}\right)}\right]$ so $f(x) = \frac{3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$, $r = \left(-\frac{x}{2}\right)$

$a=0$ Geom series: $f(x) = \frac{3}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + \left(-\frac{x}{2}\right)^n + \cdots\right]$

or $= \frac{3}{2} - \frac{3}{4}x + \frac{3}{8}x^2 - \frac{3}{16}x^3 + \cdots + \frac{3}{2}\left(-\frac{x}{2}\right)^n + \cdots$

or long division:

$$\begin{array}{r|l} & \frac{3}{2} - \frac{3}{4}x + \frac{3}{8}x^2 + \cdots \\ 2+x & 3 \\ & -(3 + \frac{3}{2}x) \\ & \quad -\frac{3}{2}x \\ & -(-\frac{3}{2}x - \frac{3}{4}x^2) \\ & \quad \frac{3}{4}x^2 \end{array}$$

④ $f(x) = \frac{x}{1-2x}$, $a = 0$ (fit to a geometric series)

$\frac{x}{1-2x}\left(\frac{-\frac{1}{2x}}{-\frac{1}{2x}}\right) = \frac{-1/2}{-\frac{1}{2x}+1} = -\frac{1}{2}\left[\frac{1}{1-\left(\frac{1}{2x}\right)}\right]$, $r = \frac{1}{2x}$

so $\frac{x}{1-2x} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2x}\right)^n = -\frac{1}{2}\left[1 + \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{8x^3} + \frac{1}{16x^4} + \cdots + \frac{1}{(2x)^n} + \cdots\right]$

$= -\frac{1}{2} - \frac{1}{4x} - \frac{1}{8x^2} - \frac{1}{16x^3} + \cdots + \frac{-1}{2(2x)^n} + \cdots$

or by long division

$$\begin{array}{r|l} & -\frac{1}{2} - \frac{1}{4x} + \cdots \\ -2x+1 & x \\ & -(x - \frac{1}{2}) \\ & \quad \frac{1}{2} \\ & -(\frac{1}{2} - \frac{1}{4x}) \end{array}$$

⑤ $f(x)=\frac{1}{4-x}$, $a=1$ (Taylor's Rule)

$$\left.\begin{array}{l} f(x)=(4-x)^{-1},\ f(1)=\frac{1}{3}=\frac{1}{3^{0+1}}=\frac{0!}{3^{0+1}} \\ f'=(-1)(4-x)^{-2}(-1),\ f'(1)=\frac{1}{9}=\frac{1}{3^{1+1}}=\frac{1!}{3^{1+1}} \\ f''=(-2)(4-x)^{-3}(-1),\ f''(1)=\frac{2}{27}=\frac{2!}{3^{2+1}} \\ f'''(x)=(-6)(-1)(4-x)^{-4},\ f'''(1)=\frac{6}{81}=\frac{3!}{3^{3+1}} \\ f^{(4)}(x)=(-24)(-1)(4-x)^{-5},\ f^{(4)}(1)=\frac{24}{243}=\frac{4!}{3^{4+1}} \end{array}\right\}\text{pattern is } \frac{n!}{3^{n+1}}$$

$$f(x)=\frac{1}{4-x}=\frac{1}{3}+\frac{1}{9}(x-1)+\frac{2/27}{2!}(x-1)^2+\frac{6/81}{3!}(x-1)^3+\frac{24/243}{4!}(x-1)^4+\dots+\frac{n!/3^{n+1}}{n!}(x-1)^n+\dots$$

$$\boxed{\frac{1}{4-x}=\frac{1}{3}+\frac{1}{9}(x-1)+\frac{1}{27}(x-2)^2+\frac{1}{81}(x-1)^3+\frac{1}{243}(x-1)^4+\dots+\frac{1}{3^{n+1}}(x-1)^n+\dots}$$

or by a recognized geom. series.

$$\frac{1}{4-x}=\frac{1}{4-(x-1)+1}=\frac{1}{3-(x-1)}\left(\frac{1/3}{1/3}\right)=\frac{1/3}{1-\frac{x-1}{3}}=\frac{1}{3}\left[\frac{1}{1-\left(\frac{x-1}{3}\right)}\right],\ r=\frac{x-1}{3}$$

$$\text{so } \frac{1}{4-x}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x-1}{3}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)\frac{(x-1)^n}{3^n}=\boxed{\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x-1)^n}$$

⑥ $f(t)=\frac{4}{1+t^2}$, $G(x)=\int_0^x f(t)\,dt$

(a) $f(t)=4\left(\frac{1}{1+t^2}\right)$ at $c=0$, so $\boxed{f(t)=4-4t^2+4t^4-4t^6+\dots+4(-1)^n(t^{2n})+\dots}$

$$\begin{array}{r|l} & 1-t^2+t^4-t^6+\dots \\ \hline 1+t^2 & 1 \\ & \underline{1+t^2} \\ & -t^2 \\ & \underline{-t^2-t^4} \\ & t^4 \\ & \underline{t^4+t^6} \\ & -t^6 \end{array}$$

(b) $G(x)=\int_0^x f(t)\,dt=4t-\frac{4}{3}t^3+\frac{4}{5}t^5-\frac{4}{7}t^7+\dots+\frac{4}{2n+1}(-1)^n(t^{2n+1})+\dots\Big|_0^x$

$=\left(4x-\frac{4}{3}x^3+\frac{4}{5}x^5-\frac{4}{7}x^7+\dots+\frac{4}{2n+1}(-1)^n(x^{2n+1})+\dots\right)-(0)$

$=\underline{4x-\frac{4}{3}x^3+\frac{4}{5}x^5-\frac{4}{7}x^7+\dots+\frac{4}{2n+1}(-1)^n(x^{2n+1})+\dots}$

(c) Interval of Convergence:

$$\lim_{n\to\infty}\left|\frac{\cancel{4}(x^{2(n+1)+1})}{2(n+1)+1}\cdot\frac{(2n+1)}{\cancel{4}x^{2n+1}}\right|=\lim_{n\to\infty}\left|\frac{x^{2n+3}\cdot(2n+1)}{(2n+3)\cdot x^{2n+1}}\right|=\lim_{n\to\infty}|x^2|\left(\frac{2n+1}{2n+3}\right)=|x^2|<1$$

$\boxed{-1<x<1}$

⑦ $f(x)=e^{-2x^2}$, $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^n}{n!}+\cdots$

(a) Replace x with $-2x^2$:

$$e^{-2x^2} = 1+(-2x^2)+\frac{(-2x^2)^2}{2}+\frac{(-2x^2)^3}{3\cdot 2}+\frac{(-2x^2)^4}{4\cdot 3\cdot 2}+\cdots+\frac{(-2x^2)^n}{n!}+\cdots$$

$$e^{-2x^2} = 1-2x^2+2x^4-\frac{4}{3}x^6+\frac{2}{3}x^8+\cdots+\frac{(-2)^n x^{2n}}{n!}+\cdots$$

Bonus term

(b) Interval of Convergence:

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{2n+1}}{(n+1)!}\cdot\frac{n!}{2^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| = 0<1 \text{ for all } x$$

so the interval of convergence is $(-\infty,\infty)$.

(c) $g(x)=1-2x^2+2x^4-\frac{4}{3}x^6$, $f(x)=e^{-2x^2}$

show $\left|e^{-2x^2}-(1-2x^2+2x^4-\frac{4}{3}x^6)\right|<0.02$ for $-0.6\le x\le 0.6$

Method 1

Let $h(x)=|f(x)-g(x)|$, $\boxed{\begin{array}{l} h(-0.6)=0.00976026<0.02 \\ h(0.6)=0.00976026<0.02 \end{array}}$

$h'(x)=0$ at $x=-0.0137071=A$ $\boxed{h(A)=0}$

* On the interval $-0.6\le x\le 0.6$, 0.00976026 is the Absolute Maximum of $h(x)$, so $|f(x)-g(x)|<0.02$ for $-0.6\le x\le 0.6$ by the EVT.

Method 2 (preferred)

By Alternating Series Remainder

$|f(x)-g(x)|\le\left|\frac{2}{3}x^8\right|$, $\left|\frac{2}{3}x^8\right|$ is maximized on $-0.6\le x\le 0.6$ at either $x=-0.6$ or $x=0.6$

So $|f(x)-g(x)|\le\left|\frac{2}{3}(.6)^8\right| = 0.01119744<0.02$

(8) $f(x) = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$, $c=0$, coefficients are $\frac{1}{(n+1)!}$

(a) $\frac{f'(0)}{1!} = \frac{1}{2!}$ ← coeff of x^1, $\frac{f^{(17)}(0)}{17!} = \frac{1}{18!}$ ← $\frac{1}{(17+1)!}$

so $f'(0) = \frac{1!}{2!} = \boxed{\frac{1}{2}}$

so $f^{(17)}(0) = \frac{17!}{18!} = \boxed{\frac{1}{18}}$

(b) Interval of Convergence:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+2} \right| = 0 \text{ for all } x$$

So $f(x)$ converges on $(-\infty, \infty)$

(c) $g(x) = x f(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots = e^x$

so $g(x) = e^x = x f(x)$

and $\boxed{f(x) = \frac{e^x}{x}}$

(9) $1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \cdots$

$= x^0 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ for $x=2$

$= e^x$ for $x=2$

so $\boxed{e^2 \approx 7.389}$

(10) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$ at $x=1$

$= \sin x$ at $x=1$

$\boxed{\text{so } \sin 1 \approx 0.84147}$

(11) $1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots + \left(\frac{1}{4}\right)^n + \cdots$

$= \frac{1}{1-x}$ for $x = \frac{1}{4}$

so $\frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \boxed{\frac{4}{3}}$

(12) $1 - \frac{100}{2!} + \frac{10000}{4!} + \cdots + \frac{(-1)^n \cdot 10^{2n}}{(2n)!} + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$ at $x=10$

$= \cos x$ at $x=10$

so $\boxed{\cos 10 \approx -0.839}$