80. For the series $\sum c_n x^n$,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x\right| < 1 \Rightarrow |x| < \left|\frac{c_n}{c_{n+1}}\right| = R$$

For the series $\sum c_n x^{2n}$,

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{2n+2}}{c_n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}x^2\right| < 1 \Rightarrow |x^2| < \left|\frac{c_n}{c_{n+1}}\right| = R \Rightarrow |x| < \sqrt{R}.$$

81. At $x = x_0 + R$, $\sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n R^n$, diverges.

At $x = x_0 - R$, $\sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(-R)^n$, converges.

Furthermore, at $x = x_0 - R$,

$$\sum_{n=0}^{\infty}\left|c_n(x - x_0)^n\right| = \sum_{n=0}^{\infty} C_n R^n, \text{ diverges.}$$

So, the series converges conditionally at $x_0 - R$.

Section 9.9 Representation of Functions by Power Series

1. (a) $$\frac{1}{4 - x} = \frac{1/4}{1 - (x/4)}$$
$$= \frac{a}{1 - r} = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)\left(\frac{x}{4}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$$

This series converges on $(-4, 4)$.

(b)
$$\begin{array}{r}
\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64} + \cdots \\
4 - x \,\overline{\big)\, 1 \qquad\qquad\quad} \\
1 - \frac{x}{4} \\
\hline
\frac{x}{4} \\
\frac{x}{4} - \frac{x^2}{16} \\
\hline
\frac{x^2}{16} \\
\frac{x^2}{16} - \frac{x^3}{64} \\
\hline
\vdots
\end{array}$$

2. (a) $$\frac{1}{2 + x} = \frac{1/2}{1 - (-x/2)} = \frac{a}{1 - r}$$
$$= \sum_{n=0}^{\infty}\frac{1}{2}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

This series converges on $(-2, 2)$.

(b)
$$\begin{array}{r}
\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots \\
2 + x \,\overline{\big)\, 1 \qquad\qquad\qquad\quad} \\
1 + \frac{x}{2} \\
\hline
-\frac{x}{2} \\
-\frac{x}{2} - \frac{x^2}{4} \\
\hline
\frac{x^2}{4} \\
\frac{x^2}{4} + \frac{x^3}{8} \\
\hline
-\frac{x^3}{8} \\
-\frac{x^3}{8} - \frac{x^4}{16} \\
\hline
\vdots
\end{array}$$

3. (a) $\dfrac{4}{3+x} = \dfrac{4/3}{1-(-x/3)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{4}{3}\left(\frac{-x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{4(-1)^n x^n}{3^{n+1}}$$

The series converges on $(-3, 3)$.

(b)
$$\begin{array}{r l}
 & \frac{4}{3} - \frac{4}{9}x + \frac{4x^2}{27} - \cdots \\
3 + x \,\big) & 4 \\
 & 4 + \frac{4}{3}x \\ \hline
 & -\frac{4}{3}x \\
 & -\frac{4}{3}x - \frac{4x^2}{9} \\ \hline
 & \frac{4x^2}{9} \\
 & \frac{4x^2}{9} + \frac{4x^3}{27} \\ \hline
 & -\frac{4x^3}{27} \\
 & \vdots
\end{array}$$

4. (a) $\dfrac{2}{5-x} = \dfrac{2/5}{1-(x/5)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{2}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \frac{2x^n}{5^{n+1}}$$

This series converges on $(-5, 5)$.

(b)
$$\begin{array}{r l}
 & \frac{2}{5} + \frac{2x}{25} + \frac{2x^2}{125} + \cdots \\
5 - x \,\big) & 2 \\
 & 2 - \frac{2}{5}x \\ \hline
 & \frac{2}{5}x \\
 & \frac{2x}{5} - \frac{2x^2}{25} \\ \hline
 & \frac{2x^2}{25} \\
 & \vdots
\end{array}$$

5. $\dfrac{1}{3-x} = \dfrac{1}{2-(x-1)} = \dfrac{1/2}{1-\left(\dfrac{x-1}{2}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(x-1)^n}{2^{n+1}}$$

Interval of convergence: $\left|\dfrac{x-1}{2}\right| < 1 \Rightarrow |x-1| < 2 \Rightarrow (-1, 3)$

6. $\dfrac{2}{6-x} = \dfrac{2}{8-(x+2)} = \dfrac{1/4}{1-\left(\dfrac{x+2}{8}\right)} = \dfrac{a}{1-r}$

$$= \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{x+2}{8}\right)^n = \sum_{n=0}^{\infty} \frac{(x+2)^n}{2^{3n+2}}$$

Interval of convergence: $\left|\dfrac{x+2}{8}\right| < 1 \Rightarrow |x+2| < 8 \Rightarrow (-10, 6)$

7. $\dfrac{1}{1-3x} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} (3x)^n$

Interval of convergence: $|3x| < 1 \Rightarrow \left(\dfrac{1}{3}, \dfrac{1}{3}\right)$

8. $\dfrac{1}{1-5x} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} (5x)^n$

Interval of convergence: $|5x| < 1 \Rightarrow \left(\dfrac{1}{5}, \dfrac{1}{5}\right)$

9. $\dfrac{5}{2x-3} = \dfrac{5}{-9+2(x+3)} = \dfrac{-5/9}{1-\dfrac{2}{9}(x+3)} = \dfrac{a}{1-r}$

$$= -\frac{5}{9}\sum_{n=0}^{\infty}\left(\frac{2}{9}(x+3)\right)^n, \quad \left|\frac{2}{9}(x+3)\right| < 1$$

$$= -5\sum_{n=0}^{\infty} \frac{2^n}{9^{n+1}}(x+3)^n$$

Interval of convergence: $\left|\dfrac{2}{9}(x+3)\right| < 1 \Rightarrow \left(-\dfrac{15}{2}, \dfrac{3}{2}\right)$

10. $\dfrac{3}{2x-1} = \dfrac{3}{3+2(x-2)} = \dfrac{1}{1+(2/3)(x-2)} = \dfrac{a}{1-r}$

$= \displaystyle\sum_{n=0}^{\infty}\left[-\frac{2}{3}(x-2)\right]^n$

$= \displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n(x-2)^n}{3^n}$

Interval of convergence: $|x-2| < \dfrac{3}{2} \Rightarrow \left(\dfrac{1}{2}, \dfrac{7}{2}\right)$

11. $\dfrac{3}{3x+4} = \dfrac{3/4}{1+\frac{3}{4}x} = \dfrac{3/4}{1-\left(-\frac{3}{4}x\right)} = \dfrac{a}{1-r}$

$= \displaystyle\sum_{n=0}^{\infty}\frac{3}{4}\left(-\frac{3}{4}x\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n 3^{n+1} x^n}{4^{n+1}}$

Interval of convergence:

$\left|-\dfrac{3}{4}x\right| < 1 \Rightarrow |3x| < 4 \Rightarrow |x| < \dfrac{4}{3} \Rightarrow \left(-\dfrac{4}{3}, \dfrac{4}{3}\right)$

12. $\dfrac{4}{3x+2} = \dfrac{4}{11+3(x-3)} = \dfrac{4/11}{1-(-3/11)(x-3)} = \dfrac{a}{1-r}$

$= \dfrac{4}{11}\displaystyle\sum_{n=0}^{\infty}\left[\frac{-3(x-3)}{11}\right]^n$

$= 4\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n(x-3)^n}{11^{n+1}}$

Interval of convergence: $\left|-\dfrac{3}{11}(x-3)\right| < 1 \Rightarrow \left(3, \dfrac{20}{3}\right)$

13. $\dfrac{4x}{x^2+2x-3} = \dfrac{3}{x+3} + \dfrac{1}{x-1}$

$= \dfrac{1}{1-(-x/3)} + \dfrac{-1}{1-x}$

$= \displaystyle\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n - \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-3)^n} - 1\right]x^n$

Interval of convergence: $\left|\dfrac{x}{3}\right| < 1$ and $|x| < 1 \Rightarrow (-1, 1)$

14. $\dfrac{3x-8}{3x^2+5x-2} = \dfrac{2}{x+2} - \dfrac{3}{3x-1}$

$= \dfrac{1}{1-(-x/2)} + \dfrac{3}{1-3x}$

$= \displaystyle\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n + 3\sum_{n=0}^{\infty}(3x)^n$

$= \displaystyle\sum_{n=0}^{\infty}\left[\left(-\frac{1}{2}\right)^n + 3^{n+1}\right]x^n$

Interval of convergence: $\left|\dfrac{x}{2}\right| < 1$ and

$|3x| < 1 \Rightarrow \left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$

15. $\dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}$

$= \displaystyle\sum_{n=0}^{\infty}\left(1+(-1)^n\right)x^n = 2\sum_{n=0}^{\infty}x^{2n}$

Interval of convergence: $|x^2| < 1$ or $(-1, 1)$ because

$\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2n+2}}{2x^{2n}}\right| = |x^2|$

16. $\dfrac{5}{5+x^2} = \dfrac{1}{1-\left(\frac{-x^2}{5}\right)} = \dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty}\left(-\frac{x^2}{5}\right)^n = \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n x^{2n}$

Interval of convergence: $\left|\dfrac{x^2}{5}\right| < 1 \Rightarrow -5 < x^2 < 5 \Rightarrow \left(-\sqrt{5}, \sqrt{5}\right)$

17. $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$

$\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty}(-1)^{2n}x^n = \sum_{n=0}^{\infty}x^n$

$h(x) = \dfrac{-2}{x^2-1} = \dfrac{1}{1+x} + \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}\left[(-1)^n + 1\right]x^n$

$= 2 + 0x + 2x^2 + 0x^3 + 2x^4 + 0x^5 + 2x^6 + \cdots = 2\displaystyle\sum_{n=0}^{\infty}x^{2n}$, $(-1, 1)$ (See Exercise 15.)

18. $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)} = \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^n - \dfrac{1}{2}\sum_{n=0}^{\infty} x^n$ (See Exercise 17.)

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left[(-1)^n - 1\right]x^n = \frac{1}{2}\left[0 - 2x + 0x^2 - 2x^3 + 0x^4 - 2x^5 + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-2)x^{2n+1} = -\sum_{n=0}^{\infty} x^{2n+1}, (-1, 1)$$

19. By taking the first derivative, you have $\dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right] = \dfrac{-1}{(x + 1)^2}$. Therefore,

$$\frac{-1}{(x + 1)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \sum_{n=1}^{\infty}(-1)^n nx^{n-1} = \sum_{n=0}^{\infty}(-1)^{n+1}(n + 1)x^n, (-1, 1).$$

20. By taking the second derivative, you have $\dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right] = \dfrac{2}{(x + 1)^3}$. Therefore,

$$\frac{2}{(x + 1)^3} = \frac{d^2}{dx^2}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right] = \frac{d}{dx}\left[\sum_{n=1}^{\infty}(-1)^n nx^{n-1}\right] = \sum_{n=2}^{\infty}(-1)^n n(n - 1)x^{n-2} = \sum_{n=0}^{\infty}(-1)^n(n + 2)(n + 1)x^n, (-1, 1).$$

21. By integrating, you have $\displaystyle\int\frac{1}{x + 1}\,dx = \ln(x + 1)$. Therefore,

$$\ln(x + 1) = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}, -1 < x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(x + 1) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}, (-1, 1].$$

22. By integrating, you have

$$\int\frac{1}{1 + x}\,dx = \ln(1 + x) + C_1 \text{ and } \int\frac{1}{1 - x}\,dx = -\ln(1 - x) + C_2.$$

$f(x) = \ln(1 - x^2) = \ln(1 + x) - \left[-\ln(1 - x)\right]$. Therefore,

$$\ln(1 - x^2) = \int\frac{1}{1 + x}\,dx - \int\frac{1}{1 - x}\,dx$$

$$= \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx - \int\left[\sum_{n=0}^{\infty}x^n\right]dx = \left[C_1 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n + 1}\right] - \left[C_2 + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n + 1}\right]$$

$$= C + \sum_{n=0}^{\infty}\frac{\left[(-1)^n - 1\right]x^{n+1}}{n + 1} = C + \sum_{n=0}^{\infty}\frac{-2x^{2n+2}}{2n + 2} = C + \sum_{n=0}^{\infty}\frac{(-1)x^{2n+2}}{n + 1}$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln(1 - x^2) = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n + 1}, (-1, 1).$$

23. $\dfrac{1}{x^2 + 1} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}, (-1, 1)$

24. $\dfrac{2x}{x^2+1} = 2x\sum_{n=0}^{\infty}(-1)^n x^{2n}$ (See Exercise 23.)

$= \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$

Because $\dfrac{d}{dx}\left(\ln\left(x^2+1\right)\right) = \dfrac{2x}{x^2+1}$, you have

$$\ln\left(x^2+1\right) = \int\left[\sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}\right]dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, -1 \le x \le 1.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

$$\ln\left(x^2+1\right) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{n+1}, [-1, 1].$$

25. Because, $\dfrac{1}{x+1} = \sum_{n=0}^{\infty}(-1)^n x^n$, you have $\dfrac{1}{4x^2+1} = \sum_{n=0}^{\infty}(-1)^n\left(4x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n(2x)^{2n}, \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

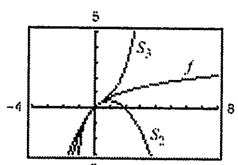
26. Because $\displaystyle\int\frac{1}{4x^2+1}\,dx = \frac{1}{2}\arctan(2x)$, you can use the result of Exercise 25 to obtain

$$\arctan(2x) = 2\int\frac{1}{4x^2+1}\,dx = 2\int\left[\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\right]dx = C + 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, -\frac{1}{2} < x < \frac{1}{2}.$$

To solve for C, let $x = 0$ and conclude that $C = 0$. Therefore,

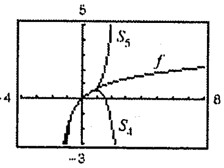
$$\arctan(2x) = 2\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{2n+1}}{2n+1}, \left(-\frac{1}{2}, \frac{1}{2}\right].$$

27. $x - \dfrac{x^2}{2} \le \ln(x+1) \le x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$S_2 = x - \dfrac{x^2}{2}$	0.000	0.180	0.320	0.420	0.480	0.500
$\ln(x+1)$	0.000	0.182	0.336	0.470	0.588	0.693
$S_3 = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$	0.000	0.183	0.341	0.492	0.651	0.833

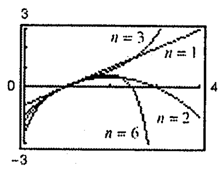
28. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} \le \ln(x+1) \le x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5}$



x	0.0	0.2	0.4	0.6	0.8	1.0
$S_4 = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}$	0.0	0.18227	0.33493	0.45960	0.54827	0.58333
$\ln(x+1)$	0.0	0.18232	0.33647	0.47000	0.58779	0.69315
$S_5 = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5}$	0.0	0.18233	0.33698	0.47515	0.61380	0.78333

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

(a)



(b) From Example 4,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{n+1}}{n+1} = \ln x,\ 0 < x \le 2,\ R = 1.$$

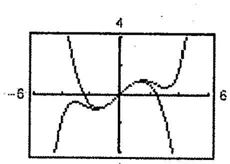
(c) $x = 0.5$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1/2)^n}{n} = \sum_{n=1}^{\infty} \frac{-(1/2)^n}{n} \approx -0.693147$$

(d) This is an approximation of $\ln\left(\dfrac{1}{2}\right)$. The error is approximately 0. [The error is less than the first omitted term, $1/(51 \cdot 2^{51}) \approx 8.7 \times 10^{-18}$.]

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(a)



(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sin x,\ R = \infty$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{2n+1}}{(2n+1)!} \approx 0.4794255386$

(d) This is an approximation of $\sin\left(\dfrac{1}{2}\right)$. The error is approximately 0.

In Exercises 31–34, $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

31. $\arctan \frac{1}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{(1/4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^{2n+1}} = \frac{1}{4} - \frac{1}{192} + \frac{1}{5120} + \cdots$

Because $\frac{1}{5120} < 0.001$, you can approximate the series by its first two terms: $\arctan \frac{1}{4} \approx \frac{1}{4} - \frac{1}{192} \approx 0.245$.

32.

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

$$\int \arctan x^2\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)} + C, C = 0$$

$$\int_0^{3/4} \arctan x^2\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{(3/4)^{4n+3}}{(4n+3)(2n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{4n+3}}{(4n+3)(2n+1)4^{4n+3}}$$

$$= \frac{27}{192} - \frac{2187}{344{,}064} + \frac{177{,}147}{230{,}686{,}720}$$

Because $177{,}147/230{,}686{,}720 < 0.001$, you can approximate the series by its first two terms: 0.134.

33.

$$\frac{\arctan x^2}{x} = \frac{1}{x}\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^2 \frac{x^{4n+1}}{2n+1}$$

$$\int \frac{\arctan x^2}{x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(4n+2)(2n+1)} + C \text{ (Note: } C = 0)$$

$$\int_0^{1/2} \frac{\arctan x^2}{x}\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+2)(2n+1)2^{4n+2}} = \frac{1}{8} - \frac{1}{1152} + \cdots$$

Because $\frac{1}{1152} < 0.001$, you can approximate the series by its first term: $\int_0^{1/2} \frac{\arctan x^2}{x}\, dx \approx 0.125$.

34.

$$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$$

$$\int x^2 \arctan x\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n+4)(2n+1)}$$

$$\int_0^{1/2} x^2 \arctan x\, dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+4)(2n+1)2^{2n+4}} = \frac{1}{64} - \frac{1}{1152} + \cdots$$

Because $\frac{1}{1152} < 0.001$, you can approximate the series by its first term: $\int_0^{1/2} x^2 \arctan x\, dx \approx 0.016$.

In Exercises 35–38, use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$.

35. $\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} nx^{n-1}, |x| < 1$

36. $\frac{x}{(1-x)^2} = x\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^n, |x| < 1$

37.

$$\frac{1+x}{(1-x)^2} = \frac{1}{(1-x)^2} + \frac{x}{(1-x)^2}$$

$$= \sum_{n=1}^{\infty} n\left(x^{n-1} + x^n\right), \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (2n+1)x^n, \quad |x| < 1$$

38. $\frac{x(1+x)}{(1-x)^2} = x\sum_{n=0}^{\infty} (2n+1)x^n = \sum_{n=0}^{\infty} (2n+1)x^{n+1}, |x| < 1$

(See Exercise 37.)

39. $P(n) = \left(\frac{1}{2}\right)^n$

$$E(n) = \sum_{n=1}^{\infty} nP(n) = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1}{2}\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1}$$
$$= \frac{1}{2}\frac{1}{\left[1-(1/2)\right]^2} = 2$$

Because the probability of obtaining a head on a single toss is $\frac{1}{2}$, it is expected that, on average, a head will be obtained in two tosses.

40. (a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n = \frac{2}{9}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^{n-1} = \frac{2}{9}\frac{1}{\left[1-(2/3)\right]^2} = 2$

(b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n = \frac{9}{100}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}$

$$= \frac{9}{100}\cdot\frac{1}{\left[1-(9/10)\right]^2} = 9$$

41. Because $\frac{1}{1+x} = \frac{1}{1-(-x)}$, substitute $(-x)$ into the geometric series.

42. Because $\frac{1}{1-x^2} = \frac{1}{1-\left(x^2\right)}$, substitute $\left(x^2\right)$ into the geometric series.

43. Because $\frac{1}{1+x} = 5\left(\frac{1}{1-(-x)}\right)$, substitute $(-x)$ into the geometric series and then multiply the series by 5.

44. Because $\ln(1-x) = -\int \frac{1}{1-x}\,dx$, integrate the series and then multiply by (-1).

45. Let $\arctan x + \arctan y = \theta$. Then,

$$\tan(\arctan x + \arctan y) = \tan\theta$$
$$\frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \tan\theta$$
$$\frac{x+y}{1-xy} = \tan\theta$$
$$\arctan\left(\frac{x+y}{1-xy}\right) = \theta.$$

Therefore,

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \text{ for } xy \neq 1.$$

46. (a) From Exercise 45, you have

$$\arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{120}{119} + \arctan\left(-\frac{1}{239}\right) = \arctan\left[\frac{(120/119) + (-1/239)}{1-(120/119)(-1/239)}\right]$$
$$= \arctan\left(\frac{28{,}561}{28{,}561}\right) = \arctan 1 = \frac{\pi}{4}$$

(b) $2\arctan\frac{1}{5} = \arctan\frac{1}{5} + \arctan\frac{1}{5} = \arctan\left[\frac{2(1/5)}{1-(1/5)^2}\right] = \arctan\frac{10}{24} = \arctan\frac{5}{12}$

$$4\arctan\frac{1}{5} = 2\arctan\frac{1}{5} + 2\arctan\frac{1}{5} = \arctan\frac{5}{12} + \arctan\frac{5}{12} = \arctan\left[\frac{2(5/12)}{1-(5/12)^2}\right] = \arctan\frac{120}{119}$$

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \frac{\pi}{4} \text{ (see part (a).)}$$

47. (a) $2\arctan\frac{1}{2} = \arctan\frac{1}{2} + \arctan\frac{1}{2} = \arctan\left[\frac{\frac{1}{2}+\frac{1}{2}}{1-(1/2)^2}\right] = \arctan\frac{4}{3}$

$$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \arctan\frac{4}{3} + \arctan\left(-\frac{1}{7}\right) = \arctan\left[\frac{(4/3)-(1/7)}{1+(4/3)(1/7)}\right] = \arctan\frac{25}{25} = \arctan 1 = \frac{\pi}{4}$$

(b) $\pi = 8\arctan\frac{1}{2} - 4\arctan\frac{1}{7} \approx 8\left[\frac{1}{2} - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} - \frac{(0.5)^7}{7}\right] - 4\left[\frac{1}{7} - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7}\right] \approx 3.14$

48. (a) $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\arctan\left[\frac{(1/2)+(1/3)}{1-(1/2)(1/3)}\right]=\arctan\left(\frac{5/6}{5/6}\right)=\frac{\pi}{4}$

(b) $$\pi = 4\left[\arctan\frac{1}{2}+\arctan\frac{1}{3}\right]$$
$$= 4\left[\frac{1}{2}-\frac{(1/2)^3}{3}+\frac{(1/2)^5}{5}-\frac{(1/2)^7}{7}\right]+4\left[\frac{1}{3}-\frac{(1/3)^3}{3}+\frac{(1/3)^5}{5}-\frac{(1/3)^7}{7}\right]\approx 4(0.4635)+4(0.3217)\approx 3.14$$

49. From Exercise 21, you have

$$\ln(x+1)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n}.$$

So, $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/2)^n}{n}$$
$$=\ln\left(\frac{1}{2}+1\right)=\ln\frac{3}{2}\approx 0.4055.$$

50. From Exercise 49, you have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^n n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1/3)^n}{n}$$
$$=\ln\left(\frac{1}{3}+1\right)=\ln\frac{4}{3}\approx 0.2877.$$

51. From Exercise 49, you have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{5^n n}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2/5)^n}{n}$$
$$=\ln\left(\frac{2}{5}+1\right)=\ln\frac{7}{5}\approx 0.3365.$$

52. From Example 5, you have $\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$
$$=\arctan 1=\frac{\pi}{4}\approx 0.7854$$

53. From Exercise 52, you have

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}(2n+1)}=\sum_{n=0}^{\infty}(-1)^n\frac{(1/2)^{2n+1}}{2n+1}$$
$$=\arctan\frac{1}{2}\approx 0.4636.$$

54. From Exercise 52, you have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{3^{2n-1}(2n-1)}=\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n+1}(2n+1)}$$
$$=\sum_{n=0}^{\infty}(-1)^n\frac{(1/3)^{2n+1}}{2n+1}$$
$$=\arctan\frac{1}{3}\approx 0.3218.$$

55. The series in Exercise 52 converges to its sum at a slower rate because its terms approach 0 at a much slower rate.

56. Because $\frac{d}{dx}\left[\sum_{n=0}^{\infty}a_n x^n\right]=\sum_{n=1}^{\infty}na_n x^{n-1}$, the radius of convergence is the same, 3.

57. Because the first series is the derivative of the second series, the second series converges for $|x+1|<4$ (and perhaps at the endpoints, $x=3$ and $x=-5$.)

58.