# 9.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Finding a Geometric Power Series** In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

1. $f(x) = \dfrac{1}{4 - x}$
2. $f(x) = \dfrac{1}{2 + x}$
3. $f(x) = \dfrac{4}{3 + x}$
4. $f(x) = \dfrac{2}{5 - x}$

**Finding a Power Series** In Exercises 5–16, find a power series for the function, centered at $c$, and determine the interval of convergence.

5. $f(x) = \dfrac{1}{3 - x}, \quad c = 1$
6. $f(x) = \dfrac{2}{6 - x}, \quad c = -2$
7. $f(x) = \dfrac{1}{1 - 3x}, \quad c = 0$
8. $h(x) = \dfrac{1}{1 - 5x}, \quad c = 0$
9. $g(x) = \dfrac{5}{2x - 3}, \quad c = -3$
10. $f(x) = \dfrac{3}{2x - 1}, \quad c = 2$
11. $f(x) = \dfrac{3}{3x + 4}, \quad c = 0$
12. $f(x) = \dfrac{4}{3x + 2}, \quad c = 3$
13. $g(x) = \dfrac{4x}{x^2 + 2x - 3}, \quad c = 0$
14. $g(x) = \dfrac{3x - 8}{3x^2 + 5x - 2}, \quad c = 0$
15. $f(x) = \dfrac{2}{1 - x^2}, \quad c = 0$
16. $f(x) = \dfrac{5}{5 + x^2}, \quad c = 0$

**Using a Power Series** In Exercises 17–26, use the power series

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

17. $h(x) = \dfrac{-2}{x^2 - 1} = \dfrac{1}{1 + x} + \dfrac{1}{1 - x}$
18. $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)}$
19. $f(x) = -\dfrac{1}{(x + 1)^2} = \dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right]$
20. $f(x) = \dfrac{2}{(x + 1)^3} = \dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right]$
21. $f(x) = \ln(x + 1) = \displaystyle\int \frac{1}{x + 1}\, dx$
22. $f(x) = \ln(1 - x^2) = \displaystyle\int \frac{1}{1 + x}\, dx - \int \frac{1}{1 - x}\, dx$
23. $g(x) = \dfrac{1}{x^2 + 1}$
24. $f(x) = \ln(x^2 + 1)$
25. $h(x) = \dfrac{1}{4x^2 + 1}$
26. $f(x) = \arctan 2x$

**Graphical and Numerical Analysis** In Exercises 27 and 28, let

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $S_n$ | | | | | | |
| $\ln(x + 1)$ | | | | | | |
| $S_{n+1}$ | | | | | | |

27. $S_2 \le \ln(x + 1) \le S_3$
28. $S_4 \le \ln(x + 1) \le S_5$

**Approximating a Sum** In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$
30. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$

**Approximating a Value** In Exercises 31–34, use the series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.

31. $\arctan \dfrac{1}{4}$
32. $\displaystyle\int_0^{3/4} \arctan x^2\, dx$
33. $\displaystyle\int_0^{1/2} \frac{\arctan x^2}{x}\, dx$
34. $\displaystyle\int_0^{1/2} x^2 \arctan x\, dx$

**Using a Power Series** In Exercises 35–38, use the power series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Find the series representation of the function and determine its interval of convergence.

35. $f(x) = \dfrac{1}{(1 - x)^2}$
36. $f(x) = \dfrac{x}{(1 - x)^2}$
37. $f(x) = \dfrac{1 + x}{(1 - x)^2}$
38. $f(x) = \dfrac{x(1 + x)}{(1 - x)^2}$

**39. Probability** A fair coin is tossed repeatedly. The probability that the first head occurs on the $n$th toss is $P(n) = \left(\frac{1}{2}\right)^n$. When this game is repeated many times, the average number of tosses required until the first head occurs is

$$E(n) = \sum_{n=1}^{\infty} nP(n).$$

(This value is called the *expected value of n*.) Use the results of Exercises 35–38 to find $E(n)$. Is the answer what you expected? Why or why not?

**40. Finding the Sum of a Series** Use the results of Exercises 35–38 to find the sum of each series.

(a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ (b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n$

**Writing** **In Exercises 41–44, explain how to use the geometric series**

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

**to find the series for the function. Do not find the series.**

**41.** $f(x) = \dfrac{1}{1+x}$ **42.** $f(x) = \dfrac{1}{1-x^2}$

**43.** $f(x) = \dfrac{5}{1+x}$ **44.** $f(x) = \ln(1-x)$

**45. Proof** Prove that

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

for $xy \neq 1$ provided the value of the left side of the equation is between $-\pi/2$ and $\pi/2$.

**46. Verifying an Identity** Use the result of Exercise 45 to verify each identity.

(a) $\arctan \dfrac{120}{119} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

(b) $4 \arctan \dfrac{1}{5} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

[*Hint:* Use Exercise 45 twice to find $4 \arctan \frac{1}{5}$. Then use part (a).]

**Approximating Pi** **In Exercises 47 and 48, (a) verify the given equation, and (b) use the equation and the series for the arctangent to approximate $\pi$ to two-decimal-place accuracy.**

**47.** $2 \arctan \dfrac{1}{2} - \arctan \dfrac{1}{7} = \dfrac{\pi}{4}$

**48.** $\arctan \dfrac{1}{2} + \arctan \dfrac{1}{3} = \dfrac{\pi}{4}$

**Finding the Sum of a Series** **In Exercises 49–54, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.**

**49.** $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{2^n n}$ **50.** $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^n n}$

**51.** $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{2^n}{5^n n}$ **52.** $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2n+1}$

**53.** $\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2^{2n+1}(2n+1)}$

**54.** $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^{2n-1}(2n-1)}$

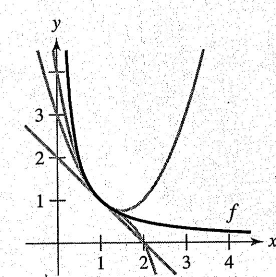
**WRITING ABOUT CONCEPTS**

**55. Using Series** One of the series in Exercises 49–54 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

**56. Radius of Convergence** The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 3. What is the radius of convergence of the series $\sum_{n=1}^{\infty} na_n x^{n-1}$? Explain.

**57. Convergence of a Power Series** The power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x+1| < 4$. What can you conclude about the series $\sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$? Explain.

**58. HOW DO YOU SEE IT?** The graphs show first-, second-, and third-degree polynomial approximations $P_1$, $P_2$, and $P_3$ of a function $f$. Label the graphs of $P_1$, $P_2$, and $P_3$. To print an enlarged copy of the graph, go to *MathGraphs.com.*




**Finding the Sum of a Series** **In Exercises 59 and 60, find the sum of the series.**

**59.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n(2n+1)}$ **60.** $\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

**61. Ramanujan and Pi** Use a graphing utility to show that

$$\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26{,}390n)}{(n!)396^{4n}} = \frac{1}{\pi}.$$

**62. Find the Error** Describe why the statement is incorrect.

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty}\left(1 + \frac{1}{5}\right)x^n$$