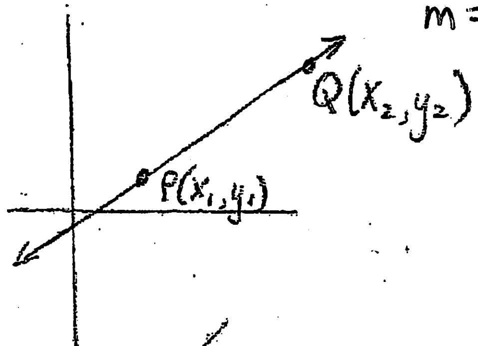
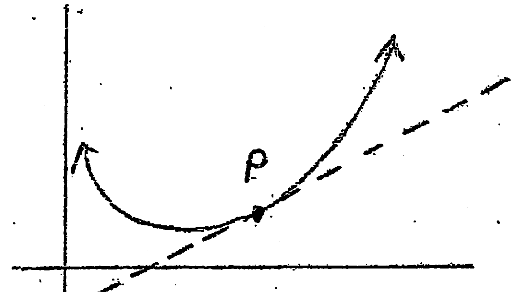
Ch. 2.1 Notes: The Derivative and Tangent Line Problem

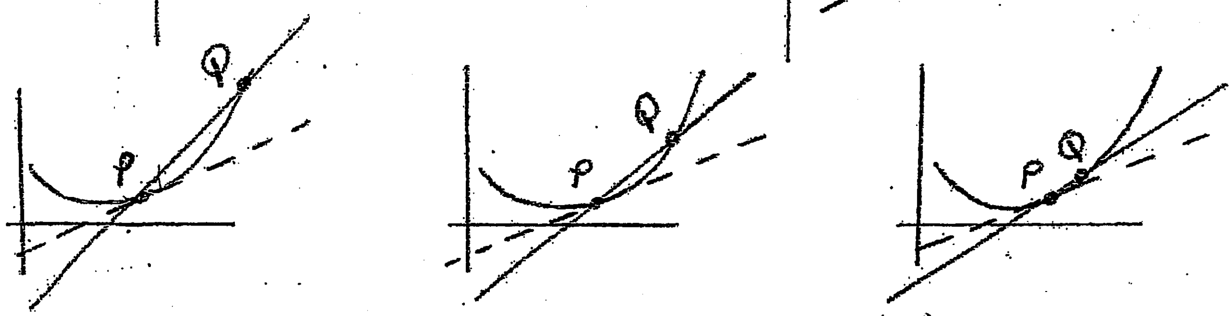
Answer Key

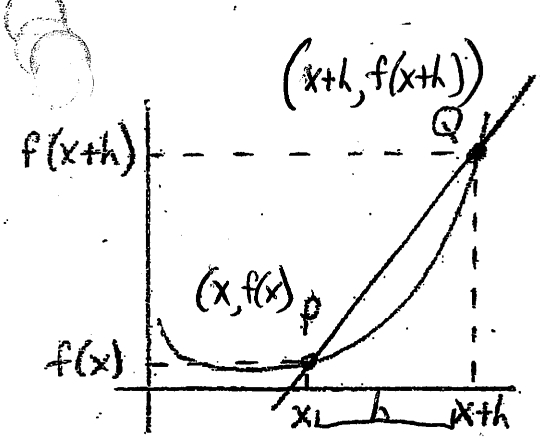
Goal: To find a formula to calculate the slope of all tangent lines to a curve.
(steepness)

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$









slope: $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$

$\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

A. General (Limit) Definition of the Derivative

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

"f prime of x": This is the notation for the derivative function

Derivative: the slope or steepness of a curve at a single point.

* The Derivative is a slope-finding formula for a curved function, where the slope is ever-changing.

B. Alternative Derivative Definition

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Ex. 1 Find the general derivative of $f(x) = x^2$. Then write the equation of the line tangent to $f(x)$ at $x=1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

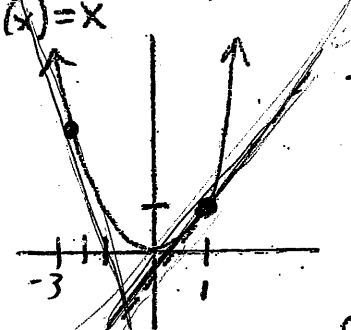
$$f'(x) = \lim_{h \to 0} \frac{(x+h)(x+h) - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + 0$$

$$\boxed{f'(x) = 2x}$$

$f(x) = x^2$



$f'(-3) = 2(-3) = -6$

$f(x) = x^2$

- $f(x)$ is the height-finding formula
- Since $f(1) = 1^2 = 1$, this tells us that when $x=1$, the height of graph has a y-value of 1

* Therefore, the derivative (slope-finding formu for $f(x) = x^2$

$f'(x) = 2x$

- $f'(x)$ is the slope-finding formula for the $f(x)$ graph
- Since $f'(1) = 2(1) = 2$, this tells us that when $x=1$ the slope of tangent line to $f(x)$ has slope of 2 (steepness)

Find Tangent-line equation:

* $y - y_1 = m(x - x_1)$ point-slope

point: $(1, 1)$

slope: 2

$y - 1 = 2(x - 1)$

Ex. 2 Find equation of tangent line to $f(x) = x^2$ at $x = -5$

$f'(x) = 2x$

$f(-5) = (-5)^2 = 25$

$f'(-5) = 2(-5) = -10$

point $(-5, 25)$

$m = -10$

$y - y_1 = m(x - x_1)$

$$\boxed{y - 25 = -10(x + 5)}$$

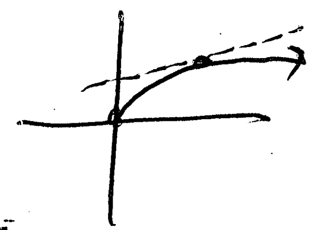
Ex. 3 (a) Find derivative of $\boxed{f(x)=\sqrt{x}}$. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

(b) Find the slope ~~of function~~ at $x=2$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{(\sqrt{x+h}+\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})}$$

$$\lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x)=\frac{1}{2\sqrt{x}}} \qquad f'(2)=\boxed{\frac{1}{2\sqrt{2}}}$$



Ex. 4 Use the alternative derivative definition to find slope of $f(x)=\sqrt{x}$ at $x=2$ $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

$c=2$ $f(2)=\sqrt{2}$

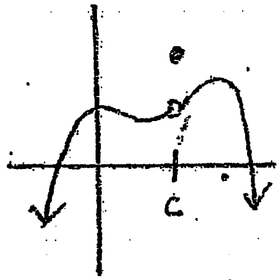
$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$f'(2)=\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

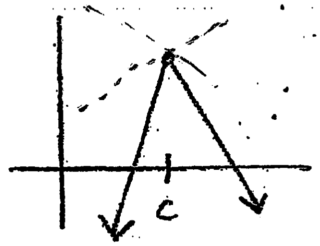
$$f'(2)=\lim_{x\to 2}\frac{(\sqrt{x}-\sqrt{2})}{x-2}\cdot\frac{(\sqrt{x}+\sqrt{2})}{\sqrt{x}+\sqrt{2}}$$

$$f'(2)=\lim_{x\to 2}\frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}=\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}}=\boxed{\frac{1}{2\sqrt{2}}}$$

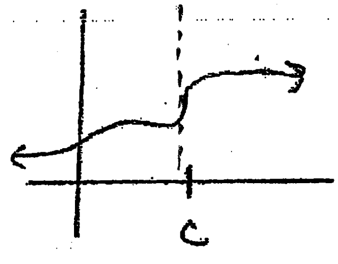
Differentiability: In order for a function to be differentiable (smooth curve) at a point, c, it must be continuous at that point, cannot contain a sharp point, cannot have vertical tangent



Graph not continuous
$f'(c)=$ DNE



Sharp point at $f(c)$
$f'(c)=$ DNE



vertical tangent at $f(c)$
$f'(c)=$ DNE

$\boxed{\text{Ex-5}}$ Use General Definition of Derivative to find

a) $f'(x)$ when $f(x) = x^2 - 5x + 2$

b) Find $f'(-3)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 5(x+h) + 2 - (x^2 - 5x + 2)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-5x} - 5h \cancel{+2} \cancel{-x^2} \cancel{+5x} \cancel{-2}}{h}$$

$$\frac{d}{dx} f(x) = 2x - 5$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 5)}{\cancel{h}} = 2x + 0 - 5 \quad \boxed{a)\ f'(x) = 2x - 5}$$

b) $f'(-3) = 2(-3) - 5 = \boxed{-11}$

$\boxed{\text{Ex. 6}}$ Alternative definition $f(x) = \sqrt{x+1}$ $c = 2$

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

$f(2) = \sqrt{2+1} = \sqrt{3}$

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x\to 2} \frac{\sqrt{x+1} - \sqrt{3}}{x - 2} \cdot \frac{(\sqrt{x+1} + \sqrt{3})}{(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x\to 2} \frac{x + 1 - 3}{(x-2)(\sqrt{x+1} + \sqrt{3})}$$

$$= \lim_{x\to 2} \frac{(\cancel{x-2})}{(\cancel{x-2})(\sqrt{x+1} + \sqrt{3})}$$

$$\lim_{x\to 2} \frac{1}{\sqrt{x+1} + \sqrt{3}} = \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$\boxed{f'(2) = \frac{1}{2\sqrt{3}}}$$

b) point: $(2, \sqrt{3})$ $m = \frac{1}{2\sqrt{3}}$

$$\boxed{y - \sqrt{3} = \frac{1}{2\sqrt{3}}(x - 2)}$$

$\boxed{Ex. 7}$ $f(x)=\frac{1}{x^2}$ find $f'(x)$ and $f'(-2)$

General definition

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}\cdot\frac{1}{h}}{h\cdot\frac{1}{h}} \quad \lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{h(x^2)(x+h)^2}$$

$$\lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}(x^2)(x+h)^2} = \frac{-2x}{(x^2)(x+0)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

$$\boxed{f'(x)=\frac{-2}{x^3}} \quad f'(-2)=\frac{-2}{(-2)^3}=\frac{-2}{-8}=\boxed{\frac{1}{4}}$$

c) Tangent line: point $(-2, \frac{1}{4})$

slope: $m=\frac{1}{4}$

$$\boxed{y-\frac{1}{4}=\frac{1}{4}(x+2)}$$

$\boxed{\text{Ex 8}}$ Alt. def $f(x) = \frac{3}{x}$ $c = 4$

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

$$f'(4) = \lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$$

$$= \lim_{x\to 4} \frac{\frac{3}{x}-\frac{3}{4}}{x-4}$$

$$= \lim_{x\to 4} \frac{\frac{12-3x}{4x}}{x-4}$$

$$f'(4) = \lim_{x\to 4} \frac{\frac{12-3x}{4x} \cdot \frac{1}{x-4}}{\cancel{x-4} \cdot \frac{1}{\cancel{x-4}}}$$

$$= \lim_{x\to 4} \frac{12-3x}{4x(x-4)}$$

$$= \lim_{x\to 4} \frac{3\cancel{(4-x)}\,-1}{4x\cancel{(x-4)}} = \lim_{x\to 4} \frac{-3}{4x}$$

$$\boxed{f'(4) = \frac{-3}{16}}$$