Name:______________________________ Period: ________

AP Calculus AB

Integrals Unit Notes Packet

Part 1

(Limit Definition of Area, Riemann Sums,
Trapezoid Approximation,
Integral Power Rule, & Trig Integrals)

Ch. 4.2a Notes

I. Sigma Notation

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$$\sum_{i=2}^{5} a_i = a_2 + a_3 + a_4 + a_5$$

______

______

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Ex. 1 $\sum_{i=2}^{4} i^2 + 1 =$

II. Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) $\sum_{i=1}^{n} c a_i = c\sum_{i=1}^{n} a_i$

Example 2

$$\sum_{i=1}^{8}(3i^2+2) =$$

Example 3

$$\sum_{i=1}^{10}(i+2)^2 =$$

Example 4

$$\sum_{k=1}^{n}\frac{1}{n}(k^2-1) =$$

②

III. Limits as n approaches infinity

*Think back about finding horizontal asymptotes

Example 5: $If\ S(n) = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$, $then\ find\ \lim_{n\to\infty} S(n)$

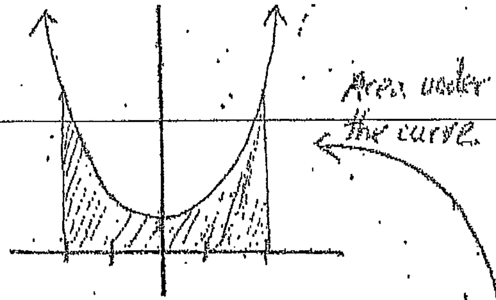
Example 6: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$

Example 7: *Find* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

4.2b - Riemann Sums

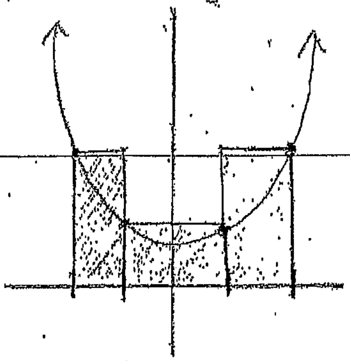
Riemann Sums - Using rectangles to estimate area of region.
(Area under a curve)

Consider the function
$f(x) = x^2 + 1 \quad [-2, 2]$



Suppose we want to estimate the <u>area of the shaded region</u> using a given number of rectangles.

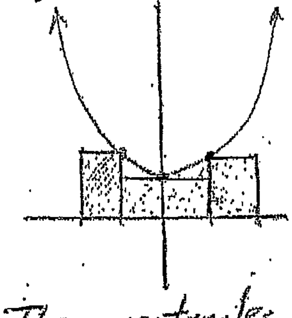
1) Upper rectangles or Circumscribed rectangles



* Using these rectangles will provide an <u>overestimation</u> of the area under the curve.

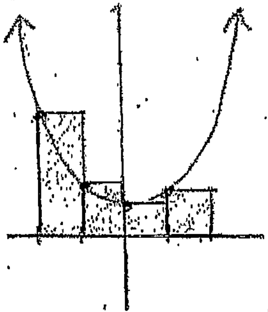
*Notice that one corner of each rectangle is on the graph. This ensures that the height of the rectangle is the same as the value of the function at the point where they connect.

2) Lower or Inscribed rectangles



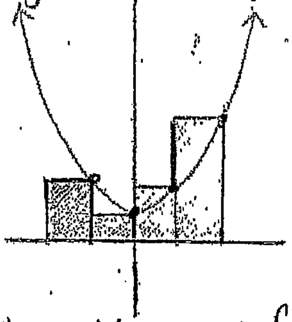
These rectangles provide an <u>underestimation</u> of the area under the curve

3) Left-handed Rectangles



The left corner of each rectangle attaches to the graph.

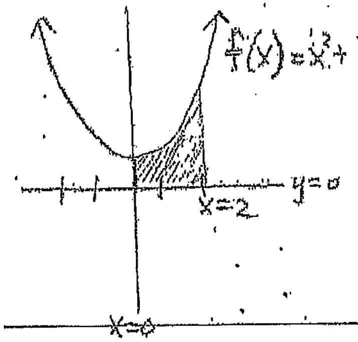
4) Right-handed Rectangles



The right corner of each rectangle attaches to the graph.

4.2b (continued)

Ex. 1 Use 4 rectangles to estimate ⓐ upper and ⓑ lower sums for the area bounded by $x=0$, $y=0$, $x=2$, and $y=x^2+1$



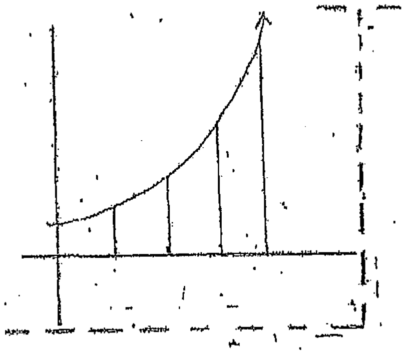
Step 1: Determine width of each rectangle.

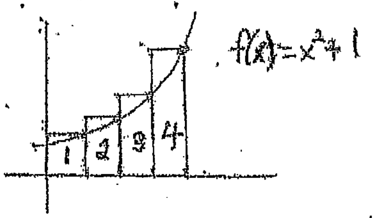
$$\boxed{\text{width} = \frac{b-a}{n}}$$

$a=$ left endpoint
$b=$ right endpoint
$n=$ number of rectangles

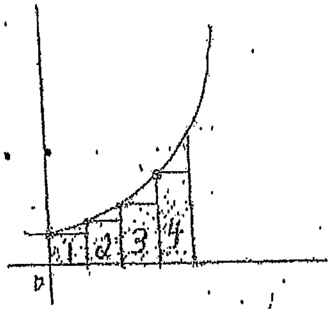
Step 2: Draw the graph. Section off each interval.

Step 3: Find sum of areas of appropriate rectangles.





b) Find lower sum

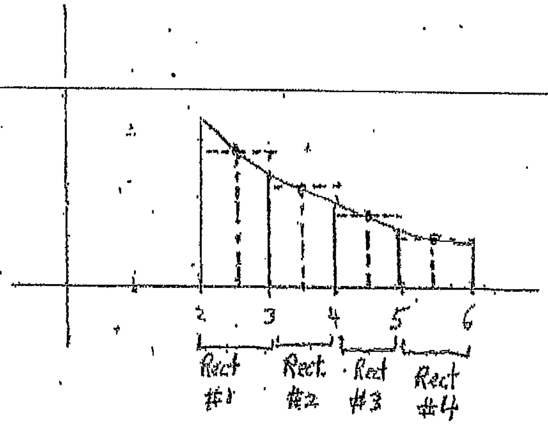


4.2b (continued)

Midpoint Rule : Similar to upper/lower sum but use the midpoint of each rectangle to calulate rectangle's height.

$\boxed{\text{Ex. 2}}$ Estimate area under curve $f(x)=\frac{8}{x^2+1}$ from $[2,6]$
Use midpoint rule with 4 subintervals.

$$\text{width} = \frac{b-a}{n} = \frac{6-2}{4} = \frac{4}{4} = 1$$



- Why would midpoint sum be a better approximation of area than upper or lower sum?

  This is because each rectangle has portions above and below the graph.

* Note: Midpoint sum is <u>not</u> the average between upper and lower sum!

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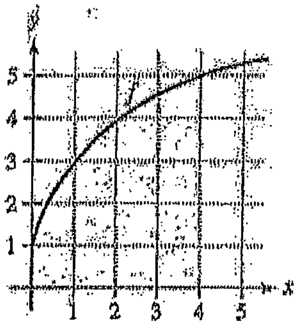
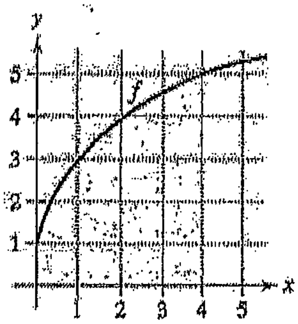
Ch. 4.2 Homework Problems

Using Upper and Lower Sums In Exercises 31 and 32, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

31.

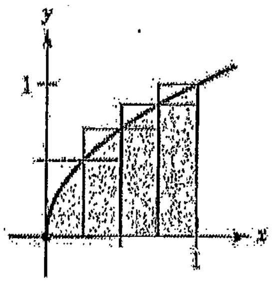


31.

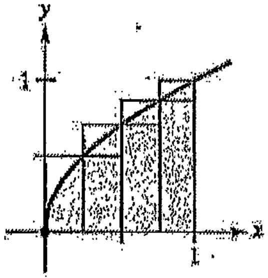


Finding Upper and Lower Sums for a Region In Exercises 33–36, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

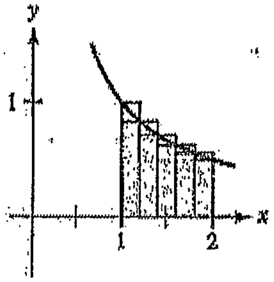
33. $y = \sqrt{x}$



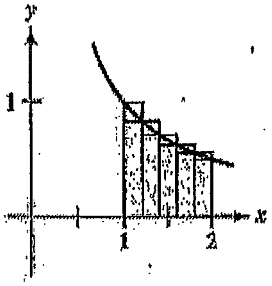
33. $y = \sqrt{x}$



35. $y = \frac{1}{x}$



35. $y = \frac{1}{x}$



Ch. 4.2 Continued

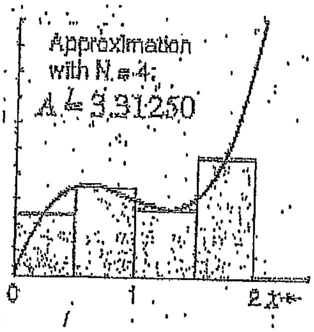
Approximating the Area of a Plane Region In Exercises 25–30, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the x-axis over the given interval.

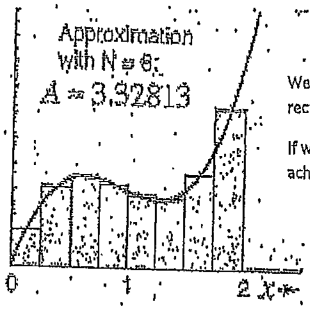
25. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles

27. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles

29. $f(x) = \cos x$, $\left[0, \dfrac{\pi}{2}\right]$, 4 rectangles

4.2a Finding Exact Area using limits





We can continually improve the Area Approximation under the curve by increasing the number of rectangles : above (n = 4) and n = 8, n = 16.

If we let n go out to infinity, (using limits) we will have something better than an approximation, we will achieve the actual area under the curve.

$$\text{Exact Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{b-a}{n}\right)\cdot f\left(a+\left(\frac{b-a}{n}\right)i\right)\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(\text{width})\cdot f\left(\underline{\text{left endpoint}} + \underline{\text{width}}\cdot i\right)$$

Memorize → "width f left plus width times i"

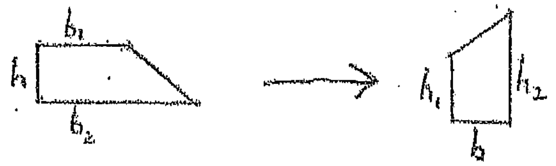
$\boxed{\text{Ex. 1}}$ Find exact area between $f(x)=4-x^2$ and x-axis from $[-2,2]$

4.6 Trapezoids

* Better approximation than inscribed, circumscribed, or midpoint rectangles.

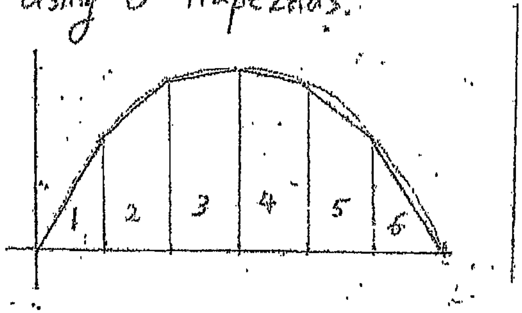
Trapezoidal Rule: Approximate Area of region using areas of trapezoids.

Review: Area of Trapezoid $= \frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$



$\boxed{\text{Area} = \frac{b}{2}(h_1+h_2)}$ or $\frac{1}{2}w(h_1+h_2)$

$\boxed{\text{Ex.1}}$ Estimate area bounded by $f(x)=6x-x^2$ and the x-axis using 6 trapezoids.



Limit Definition of Area Practice Problems WS

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{b-a}{n} \cdot f\left[a + \frac{b-a}{n} i\right]\right]$$

1) $y = 2x^2 - 3x + 2$ $[1, 3]$

2) $y = 1 - 2x - x^2$ $[-1, 4]$

⑫

Use Limit Definition of Area: $\text{Area} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \left[\frac{b-a}{n} \cdot f\left[a + \frac{b-a}{n} i\right]\right]$

1) $y = 2x^2 - 3x + 2 \qquad [1, 3]$

$\text{width} = \frac{3-1}{n} = \frac{2}{n}$

$\lim\limits_{n \to \infty} \sum \frac{2}{n} \cdot f\left[1 + \frac{2}{n} i\right]$

$\lim\limits_{n \to \infty} \sum \frac{2}{n}\left[2\left(1+\frac{2}{n}i\right)^2 - 3\left(1+\frac{2}{n}i\right) + 2\right]$

$\frac{2}{n}\left[2\left(1+\frac{2}{n}i\right)\left(1+\frac{2}{n}i\right) - 3\left(1+\frac{2}{n}i\right) + 2\right]$

$\frac{2}{n}\left[2\left(1+\frac{2}{n}i+\frac{2}{n}i+\frac{4}{n^2}i^2\right) - 3\left(1+\frac{2}{n}i\right) + 2\right]$

$\frac{2}{n}\left[2\left(1+\frac{4}{n}i+\frac{4}{n^2}i^2\right) - 3\left(1+\frac{2}{n}i\right) + 2\right]$

$\frac{2}{n}\left[2+\frac{8}{n}i+\frac{8}{n^2}i^2 - 3 - \frac{6}{n}i + 2\right]$

$\sum \frac{2}{n}\left[\frac{8}{n^2}i^2 + \frac{2}{n}i + 1\right]$

$\sum \frac{16}{n^3}i^2 + \frac{4}{n^2}i + \frac{2}{n}$

$\sum \frac{16}{n^3}i^2 + \sum \frac{4}{n^2}i + \sum \frac{2}{n}$

$\frac{16}{n^3}\boxed{\sum i^2} + \frac{4}{n^2}\boxed{\sum i} + \frac{2}{n}\boxed{\sum 1}$

$\lim\limits_{n \to \infty} \frac{16}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{4}{n^2}\left[\frac{n(n+1)}{2}\right] + \frac{2}{n}\left[n\right]$

$\lim\limits_{n \to \infty} \frac{32n^3 + \ldots}{6n^3} + \frac{4n^2 + \ldots}{2n^2} + \frac{2n}{n}$

$\frac{32}{6} + 2 + 2 = \boxed{\frac{28}{3}}$

2) $y = 1 - 2x - x^2 \qquad [-1, 4]$

$\text{width} = \frac{4-1}{n} = \frac{5}{n}$

$\lim\limits_{n \to \infty} \sum \frac{5}{n} \cdot f\left[-1 + \frac{5}{n} i\right]$

$\sum \frac{5}{n} \cdot \left[1 - 2\left(-1+\frac{5}{n}i\right) - \left(-1+\frac{5}{n}i\right)^2\right]$

$\sum \frac{5}{n}\left[1 + 2 - \frac{10}{n}i - \left(-1+\frac{5}{n}i\right)\left(-1+\frac{5}{n}i\right)\right]$

$\sum \frac{5}{n}\left[3 - \frac{10}{n}i - \left(1 - \frac{5}{n}i - \frac{5}{n}i + \frac{25}{n^2}i^2\right)\right]$

$\sum \frac{5}{n}\left[3 - \frac{10}{n}i - \left(1 - \frac{10}{n}i + \frac{25}{n^2}i^2\right)\right]$

$\sum \frac{5}{n}\left[3 - \frac{10}{n}i - 1 + \frac{10}{n}i - \frac{25}{n^2}i^2\right]$

$\sum \frac{5}{n}\left[2 - \frac{25}{n^2}i^2\right]$

$\sum \frac{10}{n} - \frac{125}{n^3}i^2 = \sum \frac{10}{n} - \sum \frac{125}{n^3}i^2$

$\frac{10}{n}\boxed{\sum 1} - \frac{125}{n^3}\boxed{\sum i^2}$

$\lim\limits_{n \to \infty} \frac{10}{n}\left[n\right] - \frac{125}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$\lim\limits_{n \to \infty} \frac{10n}{n} - \frac{250n^3 + \ldots}{6n^3}$

$10 - \frac{250}{6} = \boxed{-\frac{95}{3}}$

Riemann Sum Worksheet **Name** ____________________

For each problem sketch the graph showing the appropriate region, then approximate the area bound by the curve and the x-axis on the given interval using 6 different Riemann sums Left, Right, Upper, Lower, midpoint, Trapezoidal, using the specified number of subintervals.

1. Function : $f(x) = -(x-3)^2+20$ on Interval [0, 5] using 5 subintervals

Graph:

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

2. Function : $f(x)$ =2sinx+3 on Interval $[0, 2\pi]$using 6 subintervals

Graph:

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

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3. Function : $f(x) = \sqrt[3]{2x-1}+5$ on Interval [-2, 2] using 4 subintervals

Graph:

Left Sum	Right Sum
Lower Sum	Upper Sum
Midpoint Sum	Trapezoidal Sum

4.

x	−4	−2	0	3	6	13	20
f(x)	8	12	18	4	9	31	12

Left Sum- 6 Subintervals	Right Sum- 6 Subintervals
Lower Sum- 6 Subintervals	Upper Sum- 6 Subintervals
Midpoint Sum- 3 Subintervals	Trapezoidal Sum- 6 Subintervals

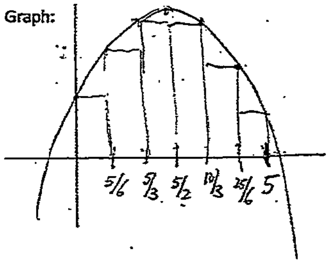
Riemann Sum Worksheet **Name** Key

For each problem sketch the graph showing the appropriate region, then approximate the area bound by the curve and the x-axis on the given interval using 6 different Riemann sums Left, Right, Upper, Lower, midpoint, Trapezoidal, using the specified number of subintervals.

1. Function : $f(x) = -(x-3)^2+20$ on Interval [0, 5] using 5 subintervals

width $h = \frac{b-a}{n} = \frac{5-0}{6} = 5/6$

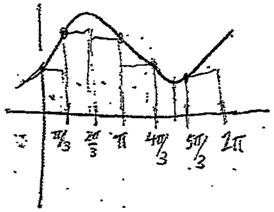
Graph:



Left Sum $5/6\left[f(0)+f(5/6)+f(5/3)+f(5/2)+f(10/3)+f(25/6)\right]$	Right Sum $5/6\left[f(5/6)+f(5/3)+f(5/2)+f(10/3)+f(25/6)+f(5)\right]$
Lower Sum $5/6\left[f(0)+f(5/6)+f(5/3)+f(10/3)+f(25/6)+f(5)\right]$	Upper Sum $5/6\left[f(5/6)+f(5/3)+f(5/2)+f(5/2)+f(10/3)+f(25/6)\right]$
Midpoint Sum $5/6\left[f(5/12)+f(5/4)+f(25/12)+f(35/12)+f(15/4)+f(55/12)\right]$	Trapezoidal Sum $\frac{w}{2}\left[h_1+2h_2+2h_3+\dots h_n\right]$ $5/6\cdot\frac{1}{2}\left[f(0)+2f(5/6)+2f(5/3)+2f(5/2)+2f(10/3)+2f(25/6)+f(5)\right]$

2. Function : $f(x) = 2\sin x+3$ on Interval $[0, 2\pi]$ using 6 subintervals $\frac{b-a}{n} = \frac{2\pi-0}{6} = \frac{\pi}{3}$

Graph:

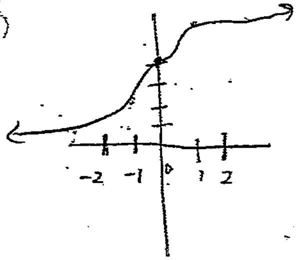


Left Sum $\frac{\pi}{3}\left[f(0)+f(\pi/3)+f(2\pi/3)+f(\pi)+f(\frac{4\pi}{3})+f(5\pi/3)\right]$	Right Sum $\frac{\pi}{3}\left[f(\pi/3)+f(\frac{2\pi}{3})+f(\pi)+f(4\pi/3)+f(5\pi/3)+f(2\pi)\right]$
Lower Sum $\frac{\pi}{3}\left[f(0)+f(\pi/3)+f(\pi)+f(\frac{4\pi}{3})+f(5\pi/3)+f(2\pi)\right]$	Upper Sum $\frac{\pi}{3}\left[f(\pi/3)+f(\frac{2\pi}{3})+f(2\pi/3)+f(\pi)+f(\frac{4\pi}{3})+f(2\pi)\right]$
Midpoint Sum $\pi/3\left[f(\pi/6)+f(\pi/2)+f(7\pi/6)+f(3\pi/2)+f(11\pi/6)\right]$	Trapezoidal Sum $\pi/3\left[f(0)+2f(\pi/3)+2f(2\pi/3)+2f(\pi)+2f(\frac{4\pi}{3})+2f(5\pi/3)+f(2\pi)\right]$

(16)

3. Function : $f(x)=\sqrt[3]{2x-1}+5$ on interval [-2, 2] using 4 subintervals $W=\frac{b-a}{n}=\frac{2-2}{4}=1$

Graph:

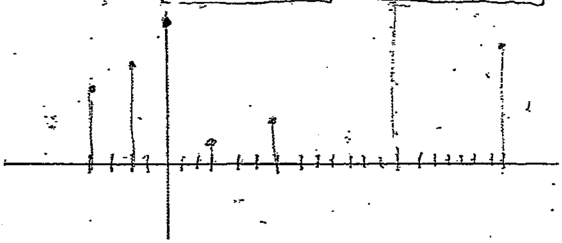


Left Sum $1\left[f(-2)+f(-1)+f(0)+f(1)\right]$	Right Sum $1\left[f(-1)+f(0)+f(1)+f(2)\right]$
Lower Sum $1\left[f(-2)+f(-1)+f(0)+f(1)\right]$	Upper Sum $1\left[f(-1)+f(0)+f(1)+f(2)\right]$
Midpoint Sum $1\left[f(-1.5)+f(-\frac{1}{2})+f(\frac{1}{2})+f(1.5)\right]$	Trapezoidal Sum $\frac{1}{2}\cdot 1\left[f(-2)+2f(-1)+2f(0)+2f(1)+f(2)\right]$

4.

1 2 3 4 5 6

x	−4	−2	0	3	6	13	20
f(x)	8	12	18	4	9	31	12



Left Sum- 6 Subintervals $2f(-4)+2(f(-2))+3f(0)+3f(3)+7f(6)+7f(13)$ $2(8)+2(12)+3(18)+3(4)+7(9)+7(31)$	Right Sum- 6 Subintervals $2f(-2)+2f(0)+3f(3)+3f(6)+7f(13)+7f(20)$
Lower Sum- 6 Subintervals $2f(-4)+2f(-2)+3f(3)+3f(3)+7f(6)+7f(20)$	Upper Sum- 6 Subintervals $2f(-2)+2f(0)+3f(0)+3f(6)+\cancel{7}f(13)+7f(13)$
Midpoint Sum- 3 Subintervals $4f(-2)+6f(3)+14f(13)$ $4(12)+6(4)+14(13)$	Trapezoidal Sum- 6 Subintervals $\frac{2}{2}\left[f(-4)+f(-2)\right]+\frac{2}{2}\left[f(-2)+f(0)\right]$ $+\frac{3}{2}\left[f(0)+f(3)\right]+\frac{3}{2}\left[f(3)+f(6)\right]$ $+\frac{7}{2}\left[f(6)+f(13)\right]+\frac{7}{2}\left[f(13)+f(20)\right]$

Non-AP Calculus **4-2, 4-6 Riemann Sums WS: Using Tables of Values**

1) Selected values of a function, *f*, are given in the table below:

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

a) Give the middle approximation with 3 subintervals for *f* on the interval [0, 20]

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 20]

c) Use left-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 9]

d) Use trapezoids to approximate the area with 2 subintervals for *f* on the interval [0, 20]

2) Selected values of a function, *f*, are given in the table below:

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

a) Give the middle approximation with 2 subintervals for *f* on the interval [1, 20]

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [3, 17]

c) Use left-handed rectangles to approximate the area with 4 subintervals for *f* on the interval [1, 12]

d) Use trapezoids to approximate the area with 3 subintervals for *f* on the interval [3, 17]

Non-AP Calculus <u>4-2, 4-6 Riemann Sums WS: Using Tables of Values</u> Key

1) Selected values of a function, *f*, are given in the table below:

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

a) Give the middle approximation with 3 subintervals for *f* on the interval [0, 20]

$\text{Area} \approx 8(2) + 4(7) + 8(6)$

$16 + 28 + 48 = \boxed{92}$

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 20]

$\text{Area} \approx 8(3) + 4(3) + 8(10)$

$24 + 12 + 80 = \boxed{116}$

c) Use left-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 9]

$\text{Area} \approx 5(4) + 3(2) + 1(3)$

$20 + 6 + 3 = \boxed{29}$

d) Use trapezoids to approximate the area with 2 subintervals for *f* on the interval [0, 20]

$\text{Area} \approx \frac{9}{2}[4+7] + \frac{11}{2}[7+10]$

$\frac{w}{2}[h_1+h_2]$ $\frac{99}{2} + \frac{187}{2} = \boxed{143}$

2) Selected values of a function, *f*, are given in the table below:

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

a) Give the middle approximation with 2 subintervals for *f* on the interval [1, 20]

$\text{Area} \approx 11(1) + 8(4)$

$11 + 32 = \boxed{43}$

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [3, 17]

$\text{Area} \approx 7(9) + 3(2) + 4(5)$

$63 + 6 + 20 = \boxed{89}$

c) Use left-handed rectangles to approximate the area with 4 subintervals for *f* on the interval [1, 12]

$\text{Area} \approx 2(3) + 4(6) + 3(1) + 2(9)$

$6 + 24 + 3 + 18 = \boxed{51}$

d) Use trapezoids to approximate the area with 3 subintervals for *f* on the interval [3, 17]

$\text{Area} \approx \frac{7}{2}[6+9] + \frac{3}{2}[9+2] + \frac{4}{2}[2+5]$

$52.5 + 16.5 + 14 = \boxed{83}$

$\frac{w}{2}[h_1+h_2]$

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Review 4.2 4.6 Formulas and Definitions:

Summation Formulas:

1) $\sum_{i=1}^{n} 1 =$

2) $\sum_{i=1}^{n} i =$

3) $\sum_{i=1}^{n} i^2 =$

4) $\sum_{i=1}^{n} i^3 =$

5) Area of Trapezoid: ______________________________

6) Width formula: ________________________________

7) Limit Definition of Area under Curve

20

Review 4.2 4.6 Formulas and Definitions:

Summation Formulas:

1) $\sum_{i=1}^{n} 1 =$ $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i =$ $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 =$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 =$ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) Area of Trapezoid: ______________________ $Area = \frac{w}{2}(h_1 + h_2)$

6) Width formula: ______________________ $width = \frac{b-a}{n}$

7) Limit Definition of Area under Curve $\lim_{n\to\infty} \sum_{i=1}^{n} (width) * f(a + width * i)$

21

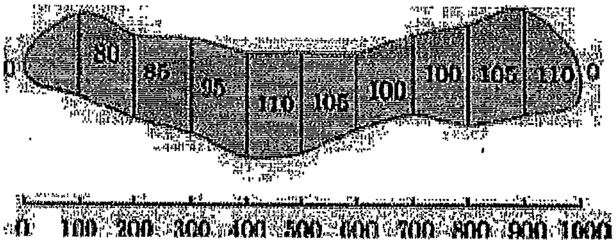
AP Calculus AB 4-2, 4-6 Quiz Review **Name**________________________

Calculators permitted.

1. Find the sum: $\sum_{i=2}^{4}\left[(i+1)^2-(2-i)^3\right]$

2. Use Sigma notation to write the sum: $\frac{2}{\sqrt[3]{5-2}}+\frac{4}{\sqrt[3]{5-4}}+\frac{6}{\sqrt[3]{5-6}}+\frac{8}{\sqrt[3]{5-8}}$

3. The width, in feet, at various points along the fairway of a hole on a golf course is given to the right. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway using trapezoids.



4. Use 3 midpoint rectangles to approximate the area of the region bounded by $f(x)=x^2+3$, the x-axis, $x=0$, and $x=5$.

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5. Use the table of values on the right to estimate the below:

x	0	4	6	7	10
$f(x)$	5	3	2	3	5

a. Use 3 left-handed rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 7]

b. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 10]

c. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [4, 10]

d. Use 3 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [0, 7]

6. Given the region bounded by $g(x) = 6 - x^2$, the x-axis, $x = -1$, and $x = 2$. Use the limit definition to find the exact area of the region.

AP Calculus AB 4-2, 4-6 Quiz Review
Calculators permitted.

Name Solution Key

1. Find the sum: $\sum_{i=2}^{4}\left[(i+1)^2-(2-i)^3\right]$

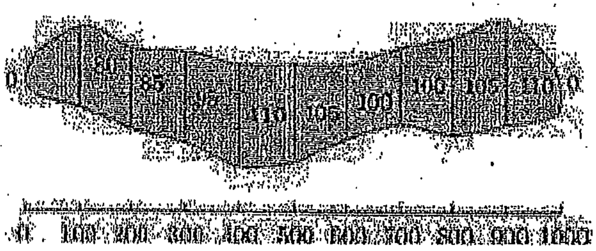
$(2+1)^2-(2-2)^3+(3+1)^2-(2-3)^3+(4+1)^2-(2-4)^3$

$9 - 0 + 16 - (-1) + 25 - (-8) = \boxed{57}$

2. Use Sigma notation to write the sum: $\frac{2}{\sqrt[3]{5-2}}+\frac{4}{\sqrt[3]{5-4}}+\frac{6}{\sqrt[3]{5-6}}+\frac{8}{\sqrt[3]{5-8}}$

$$\sum_{i=1}^{4}\frac{2i}{\sqrt[3]{5-2i}}$$

3. The width, in feet, at various points along the fairway of a hole on a golf course is given to the right. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway using trapezoids.



$A = \frac{w}{2}\left[h_1+2h_2+2h_3+2h_4+\ldots+h_n\right]$

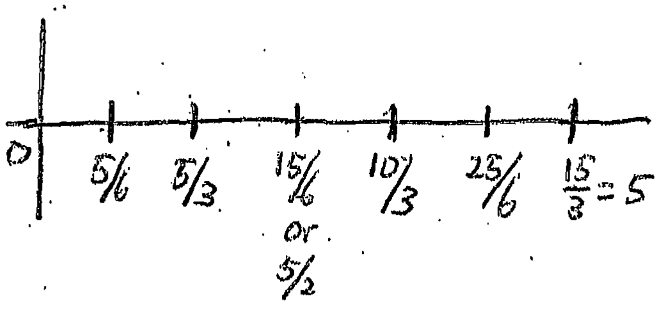
$A \approx \frac{100}{2}\left[0+2(80)+2(85)+2(95)+2(110)+2(105)+2(100)+2(100)+2(105)+2(110)+0\right]$

$A \approx 50(1780) = 89000\ ft^2$

fertilizer needed $= 89000\ ft^2 \cdot \frac{1\ lb\ fertilizer}{200\ ft^2} = \boxed{445 \text{ pounds of fertilizers}}$

4. Use 3 midpoint rectangles to approximate the area of the region bounded by $f(x)=x^2+3$, the x-axis, $x=0$, and $x=5$.

$W = \frac{b-a}{n} = \frac{5-0}{3} = \frac{5}{3}$



$A \approx \frac{5}{3}f(5/6) + \frac{5}{3}f(5/2) + \frac{5}{3}f(25/6)$

$\approx \frac{5}{3}(33.3) = \boxed{55.5}$

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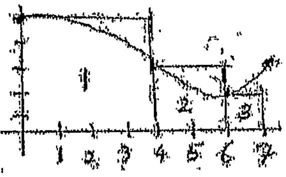
5. Use the table of values on the right to estimate the below:

x	0	4	6	7	10
$f(x)$	5	3	2	3	5

a. Use 3 left-handed rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 7]

x	0	4	6	7
f(x)	5	3	2	3



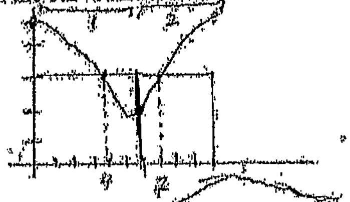
$A = 4\cdot f(0) + 2\cdot f(4) + 1\cdot f(6)$

$= 4(5) + 2(3) + 1(2)$

$= \boxed{28}$

b. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 10]

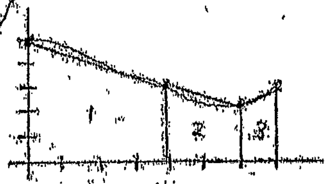
x	0	4	6	7	10
f(x)	5	3	2	3	5



$A = 6\cdot f(4) + 4\cdot f(7)$

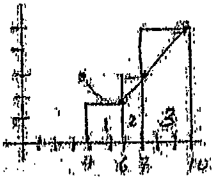
$= 6(3) + 4(3)$

$= \boxed{30}$



c. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [4, 10]

X	4	6	7	10
f(x)	3	2	3	5



$A = 2f(6) + 1f(7) + 3f(10)$

$= 2(2) + 1(3) + 3(5)$

$= 4 + 3 + 15 = \boxed{22}$

d. Use 3 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [0, 7]

X	0	4	6	7
f(x)	5	3	2	3

$A = \frac{w}{2}\left[h_1 + h_2\right]$

$\boxed{\frac{1}{2}\left[f(6)+f(7)\right]}$

$A = \frac{4}{2}\left[f(0)+f(4)\right] + \frac{2}{2}\left[f(4)+f(6)\right] + \frac{1}{2}\left[f(6) \ldots\right]$

$= 2(5+3) + 1(3+2) + \frac{1}{2}(2+3)$

$= 2(8) + 1(5) + \frac{1}{2}(5)$

$= 16 + 5 + 5/2 = \boxed{23.5 \text{ or } 47/2}$

6. Given the region bounded by g (x) = 6 – x², the x-axis, x = -1, and x = 2. Use the limit definition to find the exact area of the region.

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\cdot f\left[a + \left(\frac{b-a}{n}\right) i\right]$

$A = \lim_{n\to\infty} \sum \left(\frac{3}{n}\right)\cdot f\left[-1 + \frac{3}{n} i\right]$

$\frac{3}{n}\cdot\left[6 - \left(-1 + \frac{3}{n}i\right)^2\right] \leftarrow \left(\frac{3}{n}i - 1\right)\left(\frac{3}{n}i - 1\right)$

$\frac{9}{n^2}i^2 - \frac{6}{n}i + 1$

$\frac{3}{n}\left[6 - \left(\frac{9}{n^2}i^2 - \frac{6}{n}i + 1\right)\right]$

$\frac{3}{n}\left[6 - \frac{9}{n^2}i^2 + \frac{6}{n}i - 1\right]$

$\frac{3}{n}\left[5 - \frac{9}{n^2}i^2 + \frac{6}{n}i\right]$

$\sum \frac{15}{n} - \sum \frac{27}{n^3} i^2 + \sum \frac{18}{n^2} i$

$\frac{15}{n}\boxed{\sum 1} - \frac{27}{n^3}\boxed{\sum i^2} + \frac{18}{n^2}\boxed{\sum i}$

$\lim_{n\to\infty} \frac{15}{n}(n) - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2}\cdot\frac{n(n+1)}{2}$

$\lim_{n\to\infty} \frac{15n}{n} - \frac{54n^3 + \ldots}{6n^3} + \frac{18n^2 + \ldots}{2n^2}$

$15 - \frac{54}{6} + \frac{18}{2} = \boxed{15}$

4.2/4.6 Quiz Review WS #3

Use sigma notation to write sum

4.2, 4.6 Quiz Review #3

1) $7\left[\frac{3}{6}+4\right]+7\left[\frac{6}{6}+8\right]+7\left[\frac{9}{6}+12\right]+ \cdots 7\left[\frac{18}{6}+24\right]$

2) Use Limit Definition to find area: $h(x)=3x-x^2$ $[-1, 2]$.

3) Use 3 right-handed rectangles to approximate area of $f(x)=1+3x^2$, x-axis, $x=2$, $x=4$

4) Use 2 trapezoids to approximate area $[3, 20]$.

x	2	3	6	9	10	11	13	17	19	20	33
f(x)	8	4	1	5	6	9	3	11	4	17	19

Use sigma notation to write sum

Review #3
Key WS#3

1) $7\left[\frac{3}{6}+4\right]+7\left[\frac{6}{6}+8\right]+7\left[\frac{9}{6}+12\right]+ \_\_\_ 7\left[\frac{18}{6}+24\right]$

$$\boxed{\sum_{i=1}^{6} 7\left[\frac{3i}{6}+4i\right]}$$

2) Use Limit Definition to find area: $h(x)=3x-x^2\ [-1,2]$

$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{b-a}{n}\,f\left[a+\frac{b-a}{n}i\right]\right]$ width $=\frac{b-a}{n}=\frac{2-(-1)}{n}=\frac{3}{n}$

$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{3}{n}f\left[-1+\frac{3}{n^2}i\right]\right]$

$A=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{3}{n}\left[3(-1+\frac{3}{n}i)-(-1+\frac{3}{n}i)^2\right]\right]$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[-3+\frac{9}{n}i-\left(1-\frac{6}{n}i+\frac{9}{n^2}i^2\right)\right]$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[-4+\frac{15}{n}i-\frac{9}{n^2}i^2\right]$

$=\lim_{n\to\infty}\sum_{i=1}^{n}-\frac{12}{n}+\frac{45}{n^2}i-\frac{27}{n^3}i^2$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{-12}{n}+\sum_{i=1}^{n}\frac{45}{n^2}i-\sum_{i=1}^{n}\frac{27}{n^3}i^2$

$=\lim_{n\to\infty}-\frac{12}{n}\sum_{i=1}^{n}1+\frac{45}{n^2}\sum_{i=1}^{n}i-\frac{27}{n^3}\sum_{i=1}^{n}i^2$

$=\lim_{n\to\infty}-\frac{12}{n}(n)+\frac{45}{n^2}\left[\frac{n(n+1)}{2}\right]-\frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

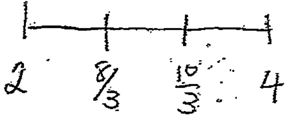
$=\lim_{n\to\infty}\frac{-12n}{n}+\frac{45n^2+45n}{2n^2}-\frac{54n^3+\dots}{6n^3}$

$=-12+\frac{45}{2}-\frac{54}{6}$

$=\boxed{\frac{3}{2}\text{ or }1.5}$

3) Use 3 right-handed rectangles to approximate area of $f(x)=1+3x^2$, x-axis, $x=2$, $x=4$

width $=\frac{b-a}{n}=\frac{4-2}{3}=\frac{2}{3}$



$A=\frac{2}{3}f(8/3)+\frac{2}{3}f\left(\frac{10}{3}\right)+\frac{2}{3}f(4)$

$A=\frac{2}{3}\left[22.33+34.33+49\right]=\boxed{70.44}$

4) Use 2 trapezoids to approximate area $[3, 20]$

x	2	3	6	9	10	11	13	17	19	20	33
f(x)	8	4	1	5	6	9	3	11	4	17	19

$A=\frac{w}{2}[h_1+h_2]$

$A=\frac{8}{2}\left[f(3)+f(11)\right]+\frac{9}{2}\left[f(11)+f(20)\right]$

$=4(4+9)+\frac{9}{2}(9+17)$

$=4(13)+4.5(26)$

$=52+117=\boxed{169}$

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AP Calculus AB 4-2, 4-6 Morning Review **Name**________________________

Calculators permitted.

1. Find the sum: $\sum_{i=1}^{3}\left[(2i+1)^2-(3i+1)^3\right]$

2. Use Sigma notation to write the sum: $\frac{7\sqrt{3}}{27}+\frac{7\sqrt{4}}{64}+\frac{7\sqrt{5}}{125}+\frac{7\sqrt{6}}{216}+\ldots\frac{7\sqrt{n}}{n^3}$

3. Use 4 middle rectangles to approximate the area of the region bounded by $f(x) = 3 + 2x^2$, the x-axis, $x = 1$, and $x = 7$.

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [1, 15]

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [5, 11]

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 15]

5. Given the region bounded by $g(x) = 3 - 2x^2$, the x-axis, $x = -1$, and $x = 1$. Use the **limit definition** to find the exact area of the region.

(28)

AP Calculus AB 4-2, 4-6 Morning Review **Name** Key

Calculators permitted.

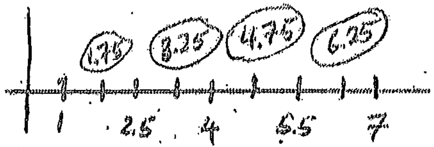
1. Find the sum: $\sum_{i=1}^{3}\left[(2i+1)^2-(3i+1)^3\right] = (2+1)^2-(3+1)^3+(4+1)^2-(6+1)^3+(6+1)^2-(9+1)^3$
$= 3^2-4^3+5^2-7^3+7^2-10^3 = \boxed{-1324}$

2. Use Sigma notation to write the sum: $\frac{7\sqrt{3}}{27}+\frac{7\sqrt{4}}{64}+\frac{7\sqrt{5}}{125}+\frac{7\sqrt{6}}{216}+\ldots\frac{7\sqrt{n}}{n^3}$

$\boxed{\sum_{i=3}^{n}\frac{7\sqrt{i}}{i^3}}$

3. Use 4 middle rectangles to approximate the area of the region bounded by $f(x) = 3+2x^2$, the x-axis, $x = 1$, and $x = 7$.

$w = \frac{b-a}{n} = \frac{7-1}{4} = \frac{3}{2}$



$A \approx (1.5)f(1.75)+1.5\cdot f(3.25)+$
$1.5\cdot f(4.75)+1.5\cdot f(6.25)$
$\approx 1.5(9.125+24.125+48.125+81.125)$
$\approx (1.5)(162.5$
$\approx \boxed{243.75}$

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [1, 15]

$5\cdot f(5)+5f(8)+4f(13)$

$5(2)+5(3)+4(6)$

$\boxed{49}$

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [5, 11]

$1f(6)+2f(8)+3f(11)$

$1(7)+2(3)+3(1)$

$\boxed{16}$

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 15] $A = \frac{w}{2}(h_1+h_2)$

$= \frac{2}{2}\left[f(6)+f(8)\right]+\frac{3}{2}\left[f(8)+f(11)\right]$
$+\frac{2}{2}\left[f(11)+f(13)\right]+\frac{2}{2}\left[f(13)+f(15)\right]$
$= \frac{2}{2}(10)+\frac{3}{2}(4)+\frac{2}{2}(7)+\frac{2}{2}(11)$

$\boxed{34}$

5. Given the region bounded by $g(x) = 3-2x^2$, the x-axis, $x = -1$, and $x = 1$. Use the **limit definition** to find the exact area of the region.

$W = \frac{1--1}{n} = \frac{2}{n}$

$A = \lim_{n\to\infty}\sum_{i=1}^{n} \text{width}\cdot f\left[a+\text{width}\cdot i\right]$

$= \frac{2}{n}\cdot f\left[-1+\frac{2}{n}i\right]$

$= \frac{2}{n}\cdot\left[3-2\left(-1+\frac{2}{n}i\right)^2\right]$

$= \frac{2}{n}\left[3-2\left(1-\frac{4}{n}i+\frac{4}{n^2}i^2\right)\right]$

$= \frac{2}{n}\left[3-2+\frac{8}{n}i-\frac{8}{n^2}i^2\right]$

$\sum \frac{2}{n}+\frac{16}{n^2}i-\frac{16}{n^3}i^2$

$\lim_{n\to\infty} \frac{2}{n}\boxed{\sum_{i=1}^{n}1}+\frac{16}{n^2}\boxed{\sum_{i=1}^{n}i}-\frac{16}{n^3}\boxed{\sum_{i=1}^{n}i^2}$

$\lim_{n\to\infty} \frac{2}{n}(n)+\frac{16}{n^2}\left[\frac{n(n+1)}{2}\right]-\frac{16}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$

$\lim_{n\to\infty} 2+\frac{16n^2}{2n^2}-\frac{16(2n^3)}{6n^3}$

$= 2+8-\frac{32}{6}$

$= \boxed{\frac{14}{3}}$

Calculus 4.1a Notes Antiderivative Formulas

If $f(x) = x^2$*, what is* $f'(x)$*?*

Using Power Rule, $\frac{d}{dx}u^n = n * u^{n-1}$*, we know that* $f'(x) = 2x$

To put the steps for Derivatives Power rule into words:

1) Bring exponent down in front of variable and ______________

2) ______________ exponent by 1

If $f'(x) = 2x$*, what steps can we take to find* $f(x)$*?*

We can "undo" the previous derivative steps:

1) ______________ 1 to the exponent

2) ______________ by the new exponent

Power Rule for Integration:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

Antidifferentiation Notation:

NOTATION: $\int 2x\,dx = x^2 + C$

Integral integrand identifies the independent variable constant of integration

Consider the below functions:

$f(x) = x^2 + 5$
$f(x) = x^2 - 13$
$f(x) = x^2 + 126$

Since we can add a constant to any of these functions and still result in the same derivative, the **antiderivative** of a function will be in the form of $f(x) + C$ to show the family of functions that share the same derivative.

The process of integration is called **antidifferentiation** or taking the indefinite integral.

The indefinite integral results in a function.

The definite integral results in a number.

30)

Integration Formulas

1. $\int u^n du = \frac{u^{n+1}}{n+1} + C$
2. $\int a\, dx = ax + C$
3. $\int \frac{1}{u}\, du = \ln|u| + C$

Important: The derivative and integral are <u>Inverse</u> operations of each other.

4) $\int f'(x)dx = f(x) + C$

5) $\frac{d}{dx}[\int f(x)\, dx] = f(x)$

Recall Power Rule Conditions:

1) Rewrite as rational exponents 2) All variables in numerator 3) Expand expression fully

Class Examples:

1. $\int 7x\, dx =$

2. $\int 7x^3 dx =$

3. $\int 2x + 3x^2 - 5x^4\, dx =$

4. $\int (3x-1)^2 dx =$

5. $\int \frac{x+1}{\sqrt{x}}\, dx =$

6. $\int \frac{3}{y\sqrt{y}}\, dy =$

7. $\int \frac{3\sqrt{x}(1-x)^2}{\sqrt[3]{x}}\, dx =$

Calculus 4.1b Notes Trig Integrals and Differential Equations

Review Derivative Trig Rules:

1) $\frac{d}{dx}\sin u =$

3) $\frac{d}{dx}\cos u =$

2) $\frac{d}{dx}\tan u =$

4) $\frac{d}{dx}\cot u =$

5) $\frac{d}{dx}\sec u =$

6) $\frac{d}{dx}\csc u =$

Integral Trig Rules:

1) $\int \sin u \, du =$

2) $\int \cos u \, du =$

3) $\int \sec^2 u \, du =$

4) $\int \csc^2 u \, du =$

5) $\int \sec u \tan u \, du =$

6) $\int \csc u \cot u \, du =$

Classwork Examples:

1. $\int \frac{\tan x}{\cos x} - \sin x \ dx$

2. $\int \frac{\sin x}{cos^2 x} dx$

3. $\int (1 + cot^2 x) dx$

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Differential Equations: These are simply equations that involve derivatives.

Steps for solving Differential equations:

1. Rewrite y′ as $\frac{dy}{dx}$
2. Separate variables on either side of equation
3. Take the integral of both sides

Solve for C if finding a specific solution/equation to the differential equation

Example 3: Suppose y′ = 2. Solve for y.

Example 4: Solve this General Differential equation. $\frac{dy}{dx} = x^3$

Example 5: Solve this Specific differential equation: y′ = 3x – 4 and the point (4, 10) is on the graph of y.

4.1b (continued) More diff. equation examples

Ex. 6 Suppose $f''(x) = 6x+4$, $f'(0) = 3$, and $f(1) = 5$.
Find $f(x)$.

※ To help distinguish the constants of integration, use "+C" for the first constant and use "+k" for the second constant of integration.

Ex. 7 Given $g''(x) = 12x+6$ and $g(0) = 4$ and $g(1) = -2$ Find $g(x)$.

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4.1, 4.2, 4.6 Formula Sheet:

Summation Formulas:

1) $\sum_{i=1}^{n} 1 = n$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

5) $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$

Area using Limit Definition

$$\lim_{n\to\infty} \sum_{i=1}^{n} (width) * f(left\, endpoint + width * i)$$

$$\text{width} = \frac{b-a}{n}$$

Trapezoid Area:

$$Area = \frac{1}{2} w(h_1 + h_2)$$

4-2, 4-6 Riemann Sums WS: Using Tables of Values

1) Selected values of a function, *f*, are given in the table below:

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

a) Give the middle approximation with 3 subintervals for *f* on the interval [0, 20]

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 20]

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

c) Use left-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [0, 9]

x	0	5	8	9	12	18	20
f(x)	4	2	3	7	3	6	10

d) Use trapezoids to approximate the area with 2 subintervals for *f* on the interval [0, 20]

㊱

2) Selected values of a function, *f*, are given in the table below:

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

a) Give the middle approximation with 2 subintervals for *f* on the interval [1, 20]

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

b) Use right-handed rectangles to approximate the area with 3 subintervals for *f* on the interval [3, 17]

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

c) Use left-handed rectangles to approximate the area with 4 subintervals for *f* on the interval [1, 12]

---

x	1	3	7	10	12	13	16	17	20
f(x)	3	6	1	9	15	2	4	5	6

d) Use trapezoids to approximate the area with 3 subintervals for *f* on the interval [3, 17]

AP Calculus AB 4-1,4-2, 4-6 Quiz Review #1

Calculators permitted.

1. Find the sum:

$$\sum_{i=2}^{4}[(i+1)^2-(2-i)^3]$$

2. Use Sigma notation to write the sum: $\frac{2}{\sqrt[3]{5-2}}+\frac{4}{\sqrt[3]{5-4}}+\frac{6}{\sqrt[3]{5-6}}+\frac{8}{\sqrt[3]{5-8}}$

3. Use 3 middle rectangles to approximate the area of the region bounded by $f(x)=x^2+3$, the x-axis, $x=1$, and $x=6$.

4. Use the table of values on the right to estimate the below:

x	0	4	6	7	10
$f(x)$	5	3	2	3	5

a. Use 3 left-handed rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 7]

b. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [0, 10]

c. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [4, 10]

d. Use 3 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [0, 7]

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5. Given the region bounded by g (x) = $6 - x^2$, the x-axis, $x = -1$, and $x = 2$. Use the limit definition to find the exact area of the region.

Find the most general antiderivative of $h(x)$. (Find $\int h(x)dx$)

6. $h(x) = 5x^4 - \pi + \frac{1}{2\sqrt{x}} + \frac{1}{3x^3}$

7. h(x) = -2cosx + 5sinx – $5cscxcotx$

8. Find the most general expression of $f(x)$ if $f''(x) = 4x^3 - 5x^2 + 3x - 6$.

---

9. Find the specific expression of $f(x)$ if $f(x) = \int g(x)dx$, $g(x) = 3x^2 - 4x$, and $f(-1) = 2$

AP Calculus AB 4-1, 4-2, 4-6 Quiz Review WS #2

Calculators permitted.

1. Find the sum:

$$\sum_{i=2}^{4}[(i+1)^2+3(2i-1)^3]$$

2. Use Sigma notation to write the sum: $\frac{5-\sqrt{2}}{1}+\frac{5-\sqrt{4}}{4}+\frac{5-\sqrt{6}}{9}+\frac{5-\sqrt{8}}{16}$

3. Use 3 left rectangles to approximate the area of the region bounded by $f(x) = 1 + 2x^2$, the x-axis, $x = 3$, and $x = 7$.

4. Use the table of values on the right to estimate the below:

x	2	5	6	8	12	13	14
f(x)	1	2	8	3	1	6	5

a. Use 2 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [5, 13]

b. Use 3 left-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [2, 14]

c. Use 2 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 14]

5. Given the region bounded by $g(x) = 3 + 2x^2$, the x-axis, $x = -2$, and $x = 1$. Use the **limit definition** to find the exact area of the region.

Find the general antiderivative of $g(x)$. (Find $\int g(x)dx$)

6. $g(x) = 3\cos x - 5\sin x + \csc x \cot x - 3\sqrt{x}$

7. $g(x) = \dfrac{2}{3(\sqrt[5]{x})} - 3x^2 - \dfrac{1}{3e^4}$

8. $g(x) = \dfrac{2x^3 - 5\sqrt{x} + 3(\sqrt[4]{x})}{x}$

9. Find the **general** expression of $f(x)$ if $f''(x) = 3x^3 + 5x^2 - x + 5$

10. Find the **specific** expression of $f(x)$ if $f'(x) = 5x^2 + 9x - 4$, $f(0) = 7$

4-1

AP Calculus AB 4-1, 4-2, 4-6 Morning Review WS #3

Calculators permitted.

1. Find the sum: $\sum_{i=1}^{3}\left[(2i+1)^{2i} + (3i+1)^3\right]$

2. Use Sigma notation to write the sum: $\frac{7\sqrt{3}}{27} + \frac{7\sqrt{4}}{64} + \frac{7\sqrt{5}}{125} + \frac{7\sqrt{6}}{216}$

3. Use 4 middle rectangles to approximate the area of the region bounded by $f(x) = 3 + 2x^2$, the x-axis, $x = 1$, and $x = 7$.

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [1, 15]

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [5, 11]

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 15]

5. Given the region bounded by $g(x) = 3 - 2x^2$, the x-axis, $x = -1$, and $x = 1$. Use the **limit definition** to find the exact area of the region.

Find the general antiderivative of $g(x)$. (Find $\int g(x)dx$)

4-2

6. $g(x) = x(2x - 1)^2$

7. $g(x) = \frac{4}{\sqrt[3]{x}} - \sqrt{x} + 3x^2 - \frac{1}{3x^4}$

8. $g(x) = \frac{x^3 - 2\sqrt{x} + \sqrt[4]{x}}{\sqrt{x}}$

9. Find the **general** expression of $f(x)$ if $f''(x) = 2x^3 + 3x^2 + x - 1$

---

10. Find the **specific** expression of $f(x)$ if $f''(x) = 12x^2 + 18x - 4$, $f'(-1) = 9$, and $f(1) = 3$

Key

AP Calculus AB 4-1, 4-2, 4-6 Morning Review WS #3

Calculators permitted.

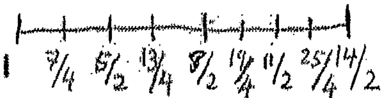
1. Find the sum: $\sum_{i=1}^{3}\left[(2i+1)^2-(3i+1)^3\right]$ $= (2(1)+1)^2-(3(1)+1)^3+(2(2)+1)^2-(3(2)+1)^3+(2(3)+1)^2-(3(3)+1)^3$

$= 3^2 - 4^3 + 5^2 - 7^3 + 7^2 - 10^3 = \boxed{-1324}$

2. Use Sigma notation to write the sum: $\frac{7\sqrt{3}}{27}+\frac{7\sqrt{4}}{64}+\frac{7\sqrt{5}}{125}+\frac{7\sqrt{6}}{216}$

3. Use 4 middle rectangles to approximate the area of the region bounded by $f(x) = 3 + 2x^2$, the x-axis, $x = 1$, and $x = 7$.

$w = \frac{7-1}{4} = \frac{6}{4} = \frac{3}{2}$



$\text{Area} \approx \frac{3}{2}\cdot f(7/4)+\frac{3}{2}\cdot f(13/4)+\frac{3}{2}\cdot f\left(\frac{19}{4}\right)+\frac{3}{2}f\left(\frac{25}{4}\right)$

$= \frac{3}{2}(9.125)+\frac{3}{2}(24.125)+\frac{3}{2}(48.125)+\frac{3}{2}(81.125) = \boxed{243.75}$

4. Use the table of values on the right to estimate the below:

x	1	5	6	8	11	13	15
f(x)	4	2	7	3	1	6	5

a. Use 3 middle rectangles with intervals indicated by the table to estimate the area between the curve and x-axis on [1, 15]

5 5 4

1	5	6	8	11	13	15
4	2	7	3	1	6	5

$5(2)+5(3)+4(6) = \boxed{49}$

b. Use 3 right-handed rectangles with intervals indicated by the table to estimate area between the curve and x-axis on [5, 11]

1 2 3

5	6	8	11
2	7	3	1

$1(7)+2(3)+3(1) = \boxed{16}$

c. Use 4 trapezoids with interval indicated by the table to estimate area between the curve and x-axis on [6, 15]

$*\text{Area} = \frac{w}{2}[h_1+h_2]$

2 3 2 2

6	8	11	13	15
7	3	1	6	5

$\frac{2}{2}[7+3]+\frac{3}{2}[3+1]+\frac{2}{2}[1+6]+\frac{2}{2}[6+5]$

$10 + \frac{3}{2}(4)+1(7)+1(11) = \boxed{34}$

5. Given the region bounded by $g(x) = 3 - 2x^2$, the x-axis, $x = -1$, and $x = 1$. Use the **limit definition** to find the exact area of the region.

$w = \frac{1-(-1)}{n} = \frac{2}{n}$

$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot f\left[-1+\frac{2}{n}i\right]$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\cdot\left[3-2\left(-1+\frac{2}{n}i\right)^2\right]$

$\sum_{i=1}^{n}\frac{2}{n}\left[3-2\left(1-\frac{4}{n}i+\frac{4}{n^2}i^2\right)\right]$

$\frac{2}{n}\left[3-2+\frac{8}{n}i-\frac{8}{n^2}i^2\right]$

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left[1+\frac{8}{n}i-\frac{8}{n^2}i^2\right]$

$\sum_{i=1}^{n}\frac{2}{n}+\frac{16}{n^2}i-\frac{16}{n^3}i^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}+\sum\frac{16}{n^2}i-\sum\frac{16}{n^3}i^2$

$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}1+\frac{16}{n^2}\sum_{i=1}^{n}i-\frac{16}{n^3}\sum_{i=1}^{n}i^2$

$\lim_{n\to\infty}\frac{2}{n}(n)+\frac{16}{n^2}\cdot\frac{n(n+1)}{2}-\frac{16}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty}\frac{2n}{n}+\frac{16n^2}{2n^2}-\frac{32n^3}{6n^3}$

$2+8-\frac{32}{6} = \boxed{\frac{14}{3}}$

Find the general antiderivative of $g(x)$. (Find $\int g(x)dx$)

$6.\ g(x) = x(2x-1)^2$

$\int x(2x-1)^2 dx$

$\int x(2x-1)(2x-1)dx$

$\int x(4x^2-4x+1)dx$

$\int 4x^3-4x^2+x\,dx$

$\frac{4x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + C$

$\boxed{x^4 - \frac{4}{3}x^3 + \frac{x^2}{2} + C}$

$7.\ g(x) = \frac{4}{\sqrt[3]{x}} - \sqrt{x} + 3x^2 - \frac{1}{3x^4}$

$\int 4x^{-1/3} - x^{1/2} + 3x^2 - \frac{1}{3}x^{-4}dx$

$4\left(\frac{x^{2/3}}{2/3}\right) - \frac{x^{3/2}}{3/2} + \frac{3x^3}{3} - \frac{1}{3}\cdot\frac{x^{-3}}{-3} + C$

$\boxed{6x^{2/3} - \frac{2}{3}x^{3/2} + x^3 + \frac{1}{9x^3} + C}$

$8.\ g(x) = \frac{x^3 - 2\sqrt{x} + \sqrt[4]{x}}{\sqrt{x}}$

$\int (x^3 - 2x^{1/2} + x^{1/4})x^{-1/2}dx$

$\int x^{5/2} - 2 + x^{-1/4}dx$

$\frac{x^{7/2}}{7/2} - 2x + \frac{x^{3/4}}{3/4} + C$

$\boxed{\frac{2}{7}x^{7/2} - 2x + \frac{4}{3}x^{3/4} + C}$

9. Find the **general** expression of $f(x)$ if $f''(x) = 2x^3 + 3x^2 + x - 1$

$f'(x) = \int 2x^3 + 3x^2 + x - 1\,dx$

$f'(x) = \frac{2x^4}{4} + \frac{3x^3}{3} + \frac{x^2}{2} - x + C$

$f(x) = \frac{1}{2}\cdot\frac{x^5}{5} + \frac{x^4}{4} + \frac{1}{2}\cdot\frac{x^3}{3} - \frac{x^2}{2} + Cx + k$

$\boxed{f(x) = \frac{1}{10}x^5 + \frac{1}{4}x^4 + \frac{1}{6}x^3 - \frac{1}{2}x^2 + Cx + k}$

10. Find the **specific** expression of $f(x)$ if $f''(x) = 12x^2 + 18x - 4$, $f'(-1) = 9$, and $f(1) = 3$

$f'(x) = \int 12x^2 + 18x - 4\,dx$

$f'(x) = \frac{12x^3}{3} + \frac{18x^2}{2} - 4x + C$

$9 = 4(-1)^3 + 9(-1)^2 - 4(-1) + C$

$9 = -4 + 9 + 4 + C$

$0 = C$

$f'(x) = 4x^3 + 9x^2 - 4x$

$f(x) = \int 4x^3 + 9x^2 - 4x\,dx$

$f(x) = \frac{4x^4}{4} + \frac{9x^3}{3} - \frac{4x^2}{2} + k$

$f(x) = x^4 + 3x^3 - 2x^2 + k$

$3 = (1)^4 + 3(1)^3 - 2(1)^2 + k$

$3 = 1 + 3 - 2 + k$

$\underline{\underline{1 = k}}$

$\boxed{f(x) = x^4 + 3x^3 - 2x^2 + 1}$

4.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Sum In Exercises 1–6, find the sum. Use the summation capabilities of a graphing utility to verify your result.

1. $\sum_{i=1}^{6}(3i+2)$
2. $\sum_{k=3}^{9}(k^2+1)$
3. $\sum_{k=0}^{4}\frac{1}{k^2+1}$
4. $\sum_{j=4}^{6}\frac{3}{j}$
5. $\sum_{k=1}^{4}c$
6. $\sum_{i=1}^{4}[(i-1)^2+(i+1)^3]$

Using Sigma Notation In Exercises 7–12, use sigma notation to write the sum.

7. $\frac{1}{5(1)}+\frac{1}{5(2)}+\frac{1}{5(3)}+\cdots+\frac{1}{5(11)}$
8. $\frac{9}{1+1}+\frac{9}{1+2}+\frac{9}{1+3}+\cdots+\frac{9}{1+14}$
9. $\left[7\left(\frac{1}{6}\right)+5\right]+\left[7\left(\frac{2}{6}\right)+5\right]+\cdots+\left[7\left(\frac{6}{6}\right)+5\right]$
10. $\left[1-\left(\frac{1}{4}\right)^2\right]+\left[1-\left(\frac{2}{4}\right)^2\right]+\cdots+\left[1-\left(\frac{4}{4}\right)^2\right]$
11. $\left[\left(\frac{2}{n}\right)^3-\frac{2}{n}\right]\left(\frac{2}{n}\right)+\cdots+\left[\left(\frac{2n}{n}\right)^3-\frac{2n}{n}\right]\left(\frac{2}{n}\right)$
12. $\left[2\left(1+\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)+\cdots+\left[2\left(1+\frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right)$

Evaluating a Sum In Exercises 13–20, use the properties of summation and Theorem 4.2 to evaluate the sum. Use the summation capabilities of a graphing utility to verify your result.

13. $\sum_{i=1}^{12}7$
14. $\sum_{i=1}^{30}-18$
15. $\sum_{i=1}^{24}4i$
16. $\sum_{i=1}^{16}(5i-4)$
17. $\sum_{i=1}^{20}(i-1)^2$
18. $\sum_{i=1}^{10}(i^2-1)$
19. $\sum_{i=1}^{15}i(i-1)^2$
20. $\sum_{i=1}^{25}(i^3-2i)$

Evaluating a Sum In Exercises 21–24, use the summation formulas to rewrite the expression without the summation notation. Use the result to find the sums for $n = 10, 100, 1000,$ and $10{,}000$.

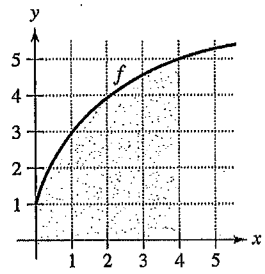
21. $\sum_{i=1}^{n}\frac{2i+1}{n^2}$
22. $\sum_{j=1}^{n}\frac{7j+4}{n^2}$
23. $\sum_{k=1}^{n}\frac{6k(k-1)}{n^3}$
24. $\sum_{i=1}^{n}\frac{2i^3-3i}{n^4}$

Approximating the Area of a Plane Region In Exercises 25–30, use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the x-axis over the given interval.

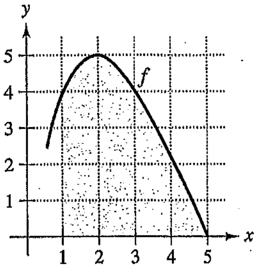
25. $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles
26. $f(x) = 9 - x$, $[2, 4]$, 6 rectangles
27. $g(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles
28. $g(x) = x^2 + 1$, $[1, 3]$, 8 rectangles
29. $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$, 4 rectangles
30. $g(x) = \sin x$, $[0, \pi]$, 6 rectangles

Using Upper and Lower Sums In Exercises 31 and 32, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

31.

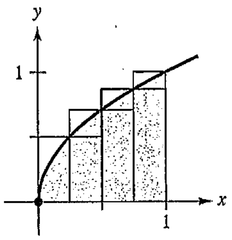


32.

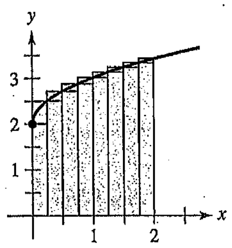


Finding Upper and Lower Sums for a Region In Exercises 33–36, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

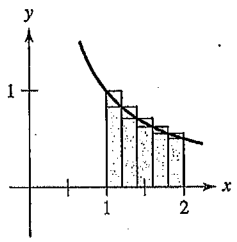
33. $y = \sqrt{x}$



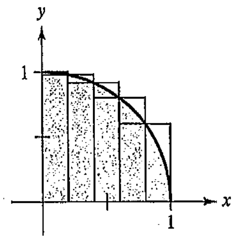
34. $y = \sqrt{x} + 2$



35. $y = \frac{1}{x}$



36. $y = \sqrt{1 - x^2}$



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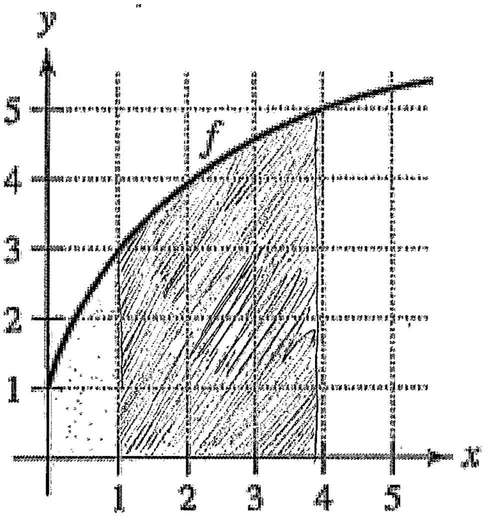
4.2 Selected HW

Using Upper and Lower Sums In Exercises 31 and 32, bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

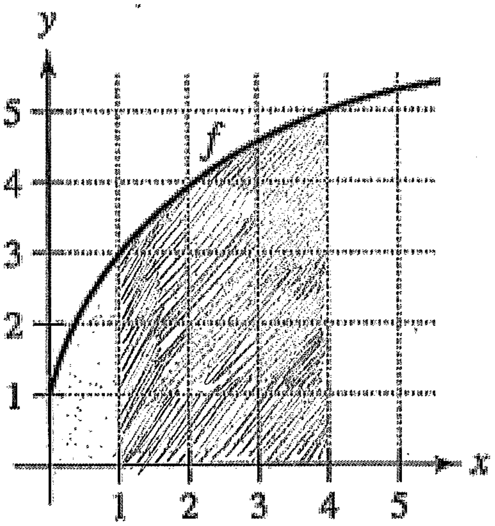
Find Upper Sum on interval [1,4]

31.



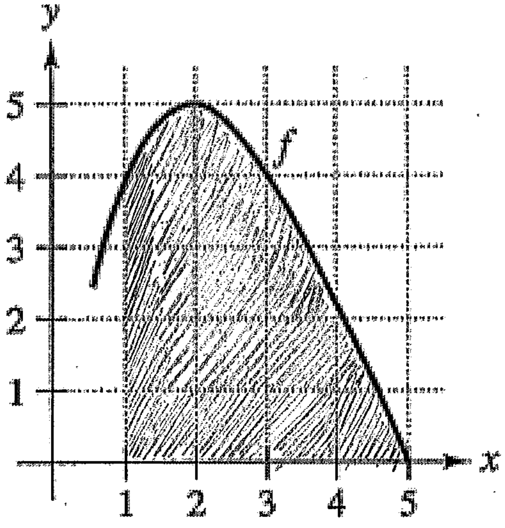
Find Lower Sum on interval [1,4]

31.



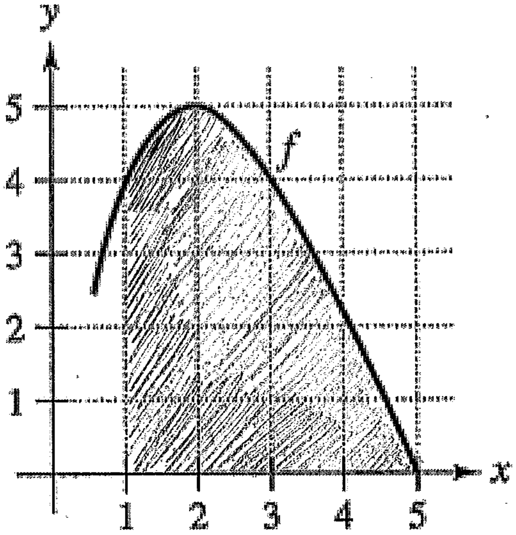
Find Upper Sum on interval [1, 5]

32.



Find Lower Sum on interval [1, 5]

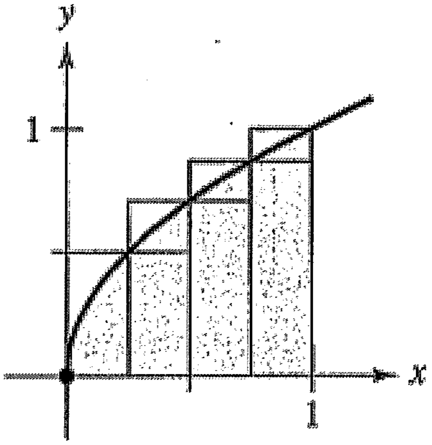
32.



Finding Upper and Lower Sums for a Region In Exercises 33–36, use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

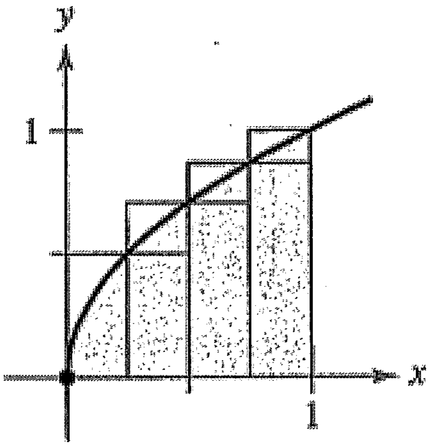
Find Upper Sum

33. $y = \sqrt{x}$



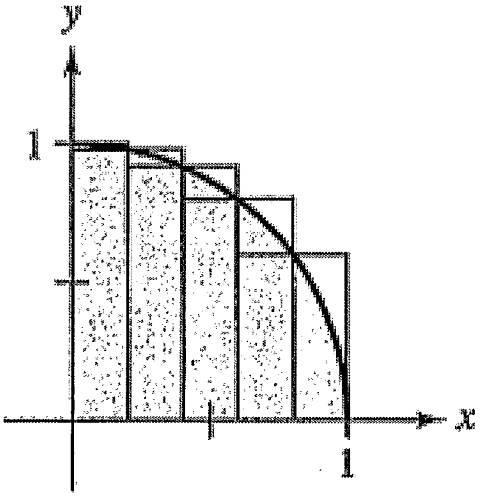
Find Lower Sum

33. $y = \sqrt{x}$



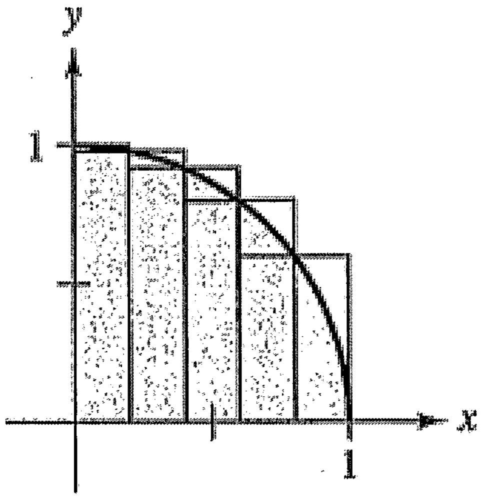
Find Upper Sum

36. $y = \sqrt{1 - x^2}$



Find Lower Sum

36. $y = \sqrt{1 - x^2}$



Finding a Limit **In Exercises 37–42, find a formula for the sum of n terms. Use the formula to find the limit as $n \to \infty$.**

37. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

38. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)\left(\frac{3}{n}\right)$

39. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

40. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)$

41. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right)$

42. $\lim_{n\to\infty} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^3\left(\frac{3}{n}\right)$

43. Numerical Reasoning Consider a triangle of area 2 bounded by the graphs of $y = x$, $y = 0$, and $x = 2$.

(a) Sketch the region.

(b) Divide the interval $[0, 2]$ into n subintervals of equal width and show that the endpoints are

$$0 < 1\left(\frac{2}{n}\right) < \cdots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 2$.

44. Numerical Reasoning Consider a trapezoid of area 4 bounded by the graphs of $y = x$, $y = 0$, $x = 1$, and $x = 3$.

(a) Sketch the region.

(b) Divide the interval $[1, 3]$ into n subintervals of equal width and show that the endpoints are

$$1 < 1 + 1\left(\frac{2}{n}\right) < \cdots < 1 + (n-1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right).$$

(c) Show that $s(n) = \sum_{i=1}^{n}\left[1 + (i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(d) Show that $S(n) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$.

(e) Complete the table.

n	5	10	50	100
$s(n)$				
$S(n)$				

(f) Show that $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n) = 4$.

Finding Area by the Limit Definition **In Exercises 45–54, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the given interval. Sketch the region.**

45. $y = -4x + 5$, $[0, 1]$

46. $y = 3x - 2$, $[2, 5]$

47. $y = x^2 + 2$, $[0, 1]$

48. $y = 3x^2 + 1$, $[0, 2]$

49. $y = 25 - x^2$, $[1, 4]$

50. $y = 4 - x^2$, $[-2, 2]$

51. $y = 27 - x^3$, $[1, 3]$

52. $y = 2x - x^3$, $[0, 1]$

53. $y = x^2 - x^3$, $[-1, 1]$

54. $y = 2x^3 - x^2$, $[1, 2]$

Finding Area by the Limit Definition **In Exercises 55–60, use the limit process to find the area of the region bounded by the graph of the function and the y-axis over the given y-interval. Sketch the region.**

55. $f(y) = 4y$, $0 \le y \le 2$

56. $g(y) = \frac{1}{2}y$, $2 \le y \le 4$

57. $f(y) = y^2$, $0 \le y \le 5$

58. $f(y) = 4y - y^2$, $1 \le y \le 2$

59. $g(y) = 4y^2 - y^3$, $1 \le y \le 3$

60. $h(y) = y^3 + 1$, $1 \le y \le 2$

Approximating Area with the Midpoint Rule **In Exercises 61–64, use the Midpoint Rule with $n = 4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.**

61. $f(x) = x^2 + 3$, $[0, 2]$

62. $f(x) = x^2 + 4x$, $[0, 4]$

63. $f(x) = \tan x$, $\left[0, \frac{\pi}{4}\right]$

64. $f(x) = \cos x$, $\left[0, \frac{\pi}{2}\right]$

WRITING ABOUT CONCEPTS

Approximation **In Exercises 65 and 66, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region, not by performing calculations.)**

65. $f(x) = 4 - x^2$, $[0, 2]$

(a) -2 (b) 6 (c) 10 (d) 3 (e) 8

66. $f(x) = \sin \frac{\pi x}{4}$, $[0, 4]$

(a) 3 (b) 1 (c) -2 (d) 8 (e) 6

67. Upper and Lower Sums In your own words and using appropriate figures, describe the methods of upper sums and lower sums in approximating the area of a region.

68. Area of a Region in the Plane Give the definition of the area of a region in the plane.

4.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Using the Trapezoidal Rule and Simpson's Rule **In Exercises 1–10, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the given value of n. Round your answer to four decimal places and compare the results with the exact value of the definite integral.**

1. $\int_0^2 x^2\,dx,\quad n = 4$

2. $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx,\quad n = 4$

3. $\int_0^2 x^3\,dx,\quad n = 4$

4. $\int_2^3 \frac{2}{x^2}\,dx,\quad n = 4$

5. $\int_1^3 x^3\,dx,\quad n = 6$

6. $\int_0^8 \sqrt[3]{x}\,dx,\quad n = 8$

7. $\int_4^9 \sqrt{x}\,dx,\quad n = 8$

8. $\int_1^4 (4 - x^2)\,dx,\quad n = 6$

9. $\int_0^1 \frac{2}{(x + 2)^2}\,dx,\quad n = 4$

10. $\int_0^2 x\sqrt{x^2 + 1}\,dx,\quad n = 4$

Using the Trapezoidal Rule and Simpson's Rule **In Exercises 11–20, approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$. Compare these results with the approximation of the integral using a graphing utility.**

11. $\int_0^2 \sqrt{1 + x^3}\,dx$

12. $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx$

13. $\int_0^1 \sqrt{x}\,\sqrt{1 - x}\,dx$

14. $\int_{\pi/2}^{\pi} \sqrt{x}\sin x\,dx$

15. $\int_0^{\sqrt{\pi/2}} \sin x^2\,dx$

16. $\int_0^{\sqrt{\pi/4}} \tan x^2\,dx$

17. $\int_3^{3.1} \cos x^2\,dx$

18. $\int_0^{\pi/2} \sqrt{1 + \sin^2 x}\,dx$

19. $\int_0^{\pi/4} x\tan x\,dx$

20. $\int_0^{\pi} f(x)\,dx,\quad f(x) = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$

WRITING ABOUT CONCEPTS

21. Polynomial Approximations The Trapezoidal Rule and Simpson's Rule yield approximations of a definite integral $\int_a^b f(x)\,dx$ based on polynomial approximations of f. What is the degree of the polynomials used for each?

22. Describing an Error Describe the size of the error when the Trapezoidal Rule is used to approximate $\int_a^b f(x)\,dx$ when $f(x)$ is a linear function. Use a graph to explain your answer.

Estimating Errors **In Exercises 23–26, use the error formulas in Theorem 4.20 to estimate the errors in approximating the integral, with $n = 4$, using (a) the Trapezoidal Rule and (b) Simpson's Rule.**

23. $\int_1^3 2x^3\,dx$

24. $\int_3^5 (5x + 2)\,dx$

25. $\int_2^4 \frac{1}{(x - 1)^2}\,dx$

26. $\int_0^{\pi} \cos x\,dx$

Estimating Errors **In Exercises 27–30, use the error formulas in Theorem 4.20 to find n such that the error in the approximation of the definite integral is less than or equal to 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.**

27. $\int_1^3 \frac{1}{x}\,dx$

28. $\int_0^1 \frac{1}{1 + x}\,dx$

29. $\int_0^2 \sqrt{x + 2}\,dx$

30. $\int_0^{\pi/2} \sin x\,dx$

Estimating Errors Using Technology **In Exercises 31–34, use a computer algebra system and the error formulas to find n such that the error in the approximation of the definite integral is less than or equal to 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.**

31. $\int_0^2 \sqrt{1 + x}\,dx$

32. $\int_0^2 (x + 1)^{2/3}\,dx$

33. $\int_0^1 \tan x^2\,dx$

34. $\int_0^1 \sin x^2\,dx$

35. Finding the Area of a Region Approximate the area of the shaded region using

(a) the Trapezoidal Rule with $n = 4$.

(b) Simpson's Rule with $n = 4$.

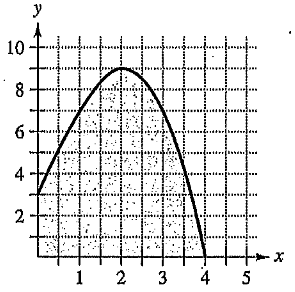


Figure for 35

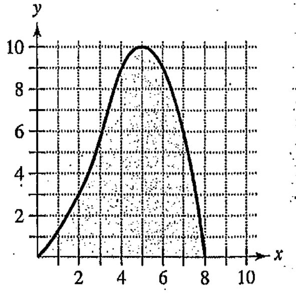


Figure for 36

36. Finding the Area of a Region Approximate the area of the shaded region using

(a) the Trapezoidal Rule with $n = 8$.

(b) Simpson's Rule with $n = 8$.

37. Area Use Simpson's Rule with $n = 14$ to approximate the area of the region bounded by the graphs of $y = \sqrt{x}\cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

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4.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Integration and Differentiation In Exercises 1 and 2, verify the statement by showing that the derivative of the right side equals the integrand of the left side.

1. $\int\left(-\frac{6}{x^4}\right)dx = \frac{2}{x^3} + C$

2. $\int\left(8x^3 + \frac{1}{2x^2}\right)dx = 2x^4 - \frac{1}{2x} + C$

Solving a Differential Equation In Exercises 3–6, find the general solution of the differential equation and check the result by differentiation.

3. $\frac{dy}{dt} = 9t^2$

4. $\frac{dy}{dt} = 5$

5. $\frac{dy}{dx} = x^{3/2}$

6. $\frac{dy}{dx} = 2x^{-3}$

Rewriting Before Integrating In Exercises 7–10, complete the table to find the indefinite integral.

Original Integral	Rewrite	Integrate	Simplify
7. $\int \sqrt[3]{x}\,dx$			
8. $\int \frac{1}{4x^2}\,dx$			
9. $\int \frac{1}{x\sqrt{x}}\,dx$			
10. $\int \frac{1}{(3x)^2}\,dx$			

Finding an Indefinite Integral In Exercises 11–32, find the indefinite integral and check the result by differentiation.

11. $\int (x + 7)\,dx$

12. $\int (13 - x)\,dx$

13. $\int (x^5 + 1)\,dx$

14. $\int (8x^3 - 9x^2 + 4)\,dx$

15. $\int (x^{3/2} + 2x + 1)\,dx$

16. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx$

17. $\int \sqrt[3]{x^2}\,dx$

18. $\int (\sqrt[4]{x^3} + 1)\,dx$

19. $\int \frac{1}{x^5}\,dx$

20. $\int \frac{3}{x^7}\,dx$

21. $\int \frac{x + 6}{\sqrt{x}}\,dx$

22. $\int \frac{x^4 - 3x^2 + 5}{x^4}\,dx$

23. $\int (x + 1)(3x - 2)\,dx$

24. $\int (4t^2 + 3)^2\,dt$

25. $\int (5\cos x + 4\sin x)\,dx$

26. $\int (t^2 - \cos t)\,dt$

27. $\int (1 - \csc t \cot t)\,dt$

28. $\int (\theta^2 + \sec^2\theta)\,d\theta$

29. $\int (\sec^2\theta - \sin\theta)\,d\theta$

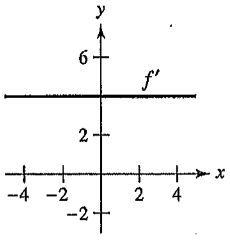
30. $\int \sec y(\tan y - \sec y)\,dy$

31. $\int (\tan^2 y + 1)\,dy$

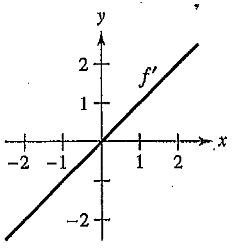
32. $\int (4x - \csc^2 x)\,dx$

Sketching a Graph In Exercises 33 and 34, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.) To print an enlarged copy of the graph, go to *MathGraphs.com.*

33.



34.



Finding a Particular Solution In Exercises 35–42, find the particular solution that satisfies the differential equation and the initial condition.

35. $f'(x) = 6x,\ f(0) = 8$

36. $g'(x) = 4x^2,\ g(-1) = 3$

37. $h'(t) = 8t^3 + 5,\ h(1) = -4$

38. $f'(s) = 10s - 12s^3,\ f(3) = 2$

39. $f''(x) = 2,\ f'(2) = 5,\ f(2) = 10$

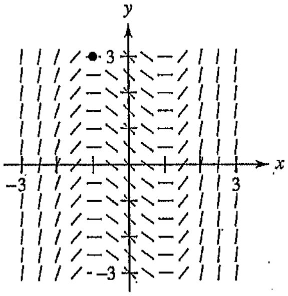
40. $f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4$

41. $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

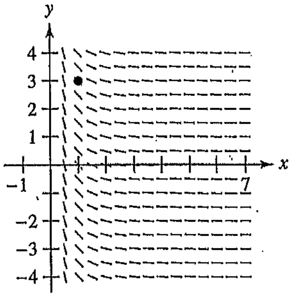
42. $f''(x) = \sin x,\ f'(0) = 1,\ f(0) = 6$

Slope Field In Exercises 43 and 44, a differential equation, a point, and a slope field are given. A *slope field* (or *direction field*) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (To print an enlarged copy of the graph, go to *MathGraphs.com.*) (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

43. $\frac{dy}{dx} = x^2 - 1,\ (-1, 3)$



44. $\frac{dy}{dx} = -\frac{1}{x^2},\ x > 0,\ (1, 3)$



Slope Field **In Exercises 45 and 46, (a) use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.**

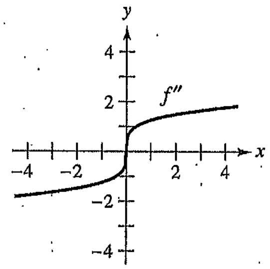
45. $\frac{dy}{dx} = 2x, (-2, -2)$ **46.** $\frac{dy}{dx} = 2\sqrt{x}, (4, 12)$

WRITING ABOUT CONCEPTS

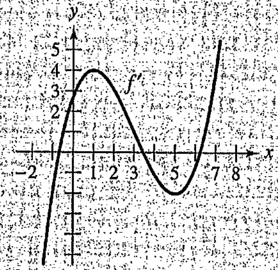
47. Antiderivatives and Indefinite Integrals What is the difference, if any, between finding the antiderivative of $f(x)$ and evaluating the integral $\int f(x)\,dx$?

48. Comparing Functions Consider $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$. What do you notice about the derivatives of $f(x)$ and $g(x)$? What can you conclude about the relationship between $f(x)$ and $g(x)$?

49. Sketching Graphs The graphs of f and f' each pass through the origin. Use the graph of f'' shown in the figure to sketch the graphs of f and f'. To print an enlarged copy of the graph, go to *MathGraphs.com.*



50. HOW DO YOU SEE IT? Use the graph of f' shown in the figure to answer the following.



(a) Approximate the slope of f at $x = 4$. Explain.

(b) Is it possible that $f(2) = -1$? Explain.

(c) Is $f(5) - f(4) > 0$? Explain.

(d) Approximate the value of x where f is maximum. Explain.

(e) Approximate any open intervals in which the graph of f is concave upward and any open intervals in which it is concave downward. Approximate the x-coordinates of any points of inflection.

51. Tree Growth An evergreen nursery usually sells a certain type of shrub after 6 years of growth and shaping. The growth rate during those 6 years is approximated by $dh/dt = 1.5t + 5$ where t is the time in years and h is the height in centimeters. The seedlings are 12 centimeters tall when planted ($t = 0$).

(a) Find the height after t years.

(b) How tall are the shrubs when they are sold?

52. Population Growth The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days $(0 \le t \le 10)$. That is,

$$\frac{dP}{dt} = k\sqrt{t}.$$

The initial size of the population is 500. After 1 day the population has grown to 600. Estimate the population after 7 days.

Vertical Motion **In Exercises 53–55, use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)**

53. A ball is thrown vertically upward from a height of 6 feet with an initial velocity of 60 feet per second. How high will the ball go?

54. With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approximately 550 feet)?

55. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

(b) At what velocity will it hit the ground?

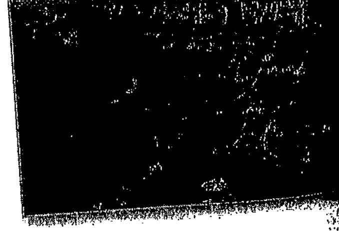
Vertical Motion **In Exercises 56–58, use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)**

56. A baseball is thrown upward from a height of 2 meters with an initial velocity of 10 meters per second. Determine its maximum height.

57. With what initial velocity must an object be thrown upward (from a height of 2 meters) to reach a maximum height of 200 meters?

58. Grand Canyon

The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Write the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon floor?



Josemaria Toscano/Shutterstock.com

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59. **Lunar Gravity** On the moon, the acceleration due to gravity is −1.6 meters per second per second. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

60. **Escape Velocity** The minimum velocity required for an object to escape Earth's gravitational pull is obtained from the solution of the equation

$$\int v\,dv = -GM\int \frac{1}{y^2}\,dy$$

where v is the velocity of the object projected from Earth, y is the distance from the center of Earth, G is the gravitational constant, and M is the mass of Earth. Show that v and y are related by the equation

$$v^2 = v_0^2 + 2GM\left(\frac{1}{y} - \frac{1}{R}\right)$$

where v_0 is the initial velocity of the object and R is the radius of Earth.

Rectilinear Motion In Exercises 61–64, consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.

61. $x(t) = t^3 - 6t^2 + 9t - 2, \quad 0 \le t \le 5$
    - (a) Find the velocity and acceleration of the particle.
    - (b) Find the open t-intervals on which the particle is moving to the right.
    - (c) Find the velocity of the particle when the acceleration is 0.

62. Repeat Exercise 61 for the position function

$$x(t) = (t-1)(t-3)^2, \quad 0 \le t \le 5.$$

63. A particle moves along the x-axis at a velocity of $v(t) = 1/\sqrt{t}$, $t > 0$. At time $t = 1$, its position is $x = 4$. Find the acceleration and position functions for the particle.

64. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position is $x = 3$.
    - (a) Find the velocity and position functions for the particle.
    - (b) Find the values of t for which the particle is at rest.

65. **Acceleration** The maker of an automobile advertises that it takes 13 seconds to accelerate from 25 kilometers per hour to 80 kilometers per hour. Assume the acceleration is constant.
    - (a) Find the acceleration in meters per second per second.
    - (b) Find the distance the car travels during the 13 seconds.

66. **Deceleration** A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.
    - (a) How far has the car moved when its speed has been reduced to 30 miles per hour?
    - (b) How far has the car moved when its speed has been reduced to 15 miles per hour?
    - (c) Draw the real number line from 0 to 132. Plot the points found in parts (a) and (b). What can you conclude?

67. **Acceleration** At the instant the traffic light turns green, a car that has been waiting at an intersection starts with a constant acceleration of 6 feet per second per second. At the same instant, a truck traveling with a constant velocity of 30 feet per second passes the car.
    - (a) How far beyond its starting point will the car pass the truck?
    - (b) How fast will the car be traveling when it passes the truck?

68. **Acceleration** Assume that a fully loaded plane starting from rest has a constant acceleration while moving down a runway. The plane requires 0.7 mile of runway and a speed of 160 miles per hour in order to lift off. What is the plane's acceleration?

True or False? **In Exercises 69–74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. The antiderivative of $f(x)$ is unique.

70. Each antiderivative of an nth-degree polynomial function is an $(n+1)$th-degree polynomial function.

71. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.

72. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then

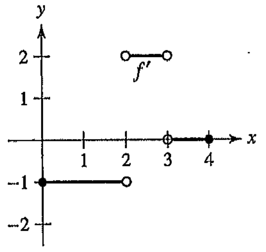
$$F(x) = G(x) + C.$$

73. If $f'(x) = g(x)$, then $\int g(x)\,dx = f(x) + C$.

74. $\int f(x)g(x)\,dx = \int f(x)\,dx \int g(x)\,dx$

75. **Horizontal Tangent** Find a function f such that the graph of f has a horizontal tangent at $(2, 0)$ and $f''(x) = 2x$.

76. **Finding a Function** The graph of f' is shown. Find and sketch the graph of f given that f is continuous and $f(0) = 1$.



77. **Proof** Let $s(x)$ and $c(x)$ be two functions satisfying $s'(x) = c(x)$ and $c'(x) = -s(x)$ for all x. If $s(0) = 0$ and $c(0) = 1$, prove that $[s(x)]^2 + [c(x)]^2 = 1$.

PUTNAM EXAM CHALLENGE

78. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y) \quad \text{and}$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

This problem was composed by the Committee on the Putnam Prize Competition.