Name: ______________________ Period: __________

AB Calculus

Unit 4

Applications of Differentiation

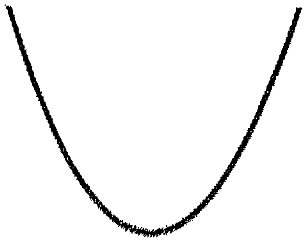
Part 1

(Related Rates, Linear Approximation, L'Hopital's Rule)

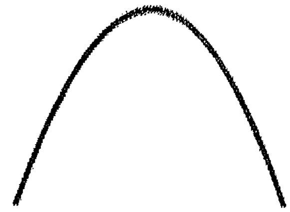
AP Calculus – 4.2 Notes Linear Approximation and rates of change other than motion

The tangent line of the function $f(x)$ at $x = a$ can give you an approximate value of $f(x)$ for points close to $x = a$.

Concave UP with a Tangent Line



Concave DOWN with a Tangent Line



1. f is concave up on its domain and $f(4) = 5$ and $f'(4) = 3$.
   a. What is the estimate for $f(3.8)$ using the local linear approximation for f at $x = 4$?

   b. Is it an underestimate or overestimate? Explain.

2. The function $f(x) = 5x - 2x^3 - 2$ is concave down at $x = 1$?.
   a. Find the tangent line of f at $x = 1$.

   b. What is the estimate for $f(1.1)$ using the local linear approximation for f at $x = 1$?

   c. Is it an underestimate or overestimate? Explain.

3. Consider the differential equation $\frac{dy}{dx} = e^y(2x^2 - 5x)$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 0$.
   a. Write an equation for the line tangent to the graph of f at the point (2,0).

   b. Use the tangent line to approximate $f(2.2)$.

(2)

Linear Approximation Practice Problems

Use the given information and a linear approximation to approximate the value of the function at c

Linear approximation steps:

i) Find the tangent line equation   ii) plug in decimal into linear equation.

1) $f(2) = 8;\ f'(2) = -3;\ c = 2.06$

2) $f(-4) = 3;\ f'(-4) = 2;\ c = -3.6$

3) $f(-1) = 0;\ f'(-1) = \frac{3}{2};\ c = -1.1$

4) $f(5) = \frac{1}{2};\ f'(5) = -3;\ c = 5.2$

Linear approximation steps:

i) Find the tangent line equation using nearest integer

ii) plug in decimal into linear equation.

5) $Given\ f(x) = 2x^3 - 4x + 1 \quad Use\ Linear\ approximation\ to\ approximate\ f(1.02)$

6) $Given\ f(x) = \frac{2x^3-1}{3-x}\ Use\ Linear\ approximation\ to\ approximate\ f(1.01)$

7) $Given\ f(x) = \sqrt[3]{3x+2} \quad Use\ Linear\ approximation\ to\ approximate\ f(2.1)$

4

Calculus Notes Ch. 4.3 - Related Rates

Related Rates: Problems involving finding the rate of change for a variable with respect to time

This is also an application of implicit differentiation: Finding derivatives of variables with respect to time t.

Related Rates Steps:

1. Write what you are given
2. Write what you are trying to find
3. Write an algebraic or geometric equation relating the variables (needs to be in terms of the rates that you are either given or are trying to find)
4. Differentiate equation with respect to time t
5. Substitute and solve

*Important Note: Remember that when the item is getting bigger, the rate is positive
If the item is getting smaller, the rate is negative – regardless of direction

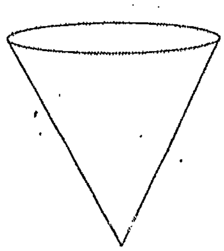
Example 1: The sides of a square are increasing at a rate of 5 cm/min... How fast is the area increasing when the sides measure 15 cm in length?

Example 2: A 25 foot ladder is leaning against a vertical wall. The bottom of the ladder is being pulled horizontally away from the wall at the constant rate of 3 ft / sec. a) How fast is the top of the ladder moving and in what direction when the bottom of the ladder is 15 ft from the wall? b) at what rate is the area changing when the bottom of the ladder is 15 ft from the wall?

Example 3: A spherical balloon is being inflated at a rate of 10 cubic centimeters per second. Find the rate of change of the surface area of the balloon at the moment when the surface area is 64π.

Example 4: Joe is standing 6 miles straight east of Moe. If Joe walks straight north at 3 mph while Moe walks straight south at 1 mph, at what rate is the distance between them changing after 2 hours?

Example 5: A conical tank (vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.



6

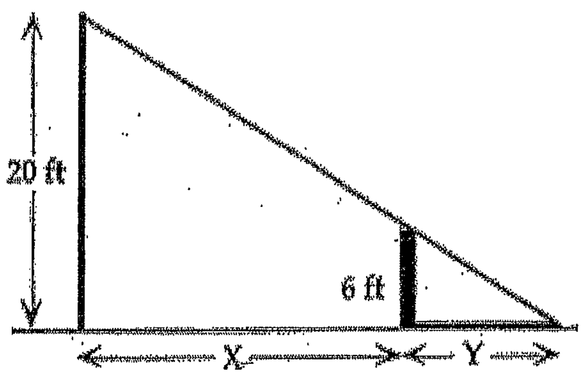
Related Rates Notes 2 - Similar Triangles and Shadow Problems

Example 1:

A man who is 6 feet tall is walking away from a lamp post at a rate of 5 feet per minute.

The lamp post is 20 feet tall. The person casts a shadow on the ground in front of them.

a) How fast is the shadow growing when the person is 30 feet from the lamp post?
b) How fast is the tip of the shadow moving when the person is 30 ft from the lamp post?



Notes:

1) $\frac{dx}{dt} = rate\ of\ person\ walking$

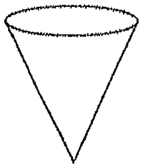
2) $\frac{dy}{dt} = rate\ of\ change\ of\ shadow\ length$

3) $\frac{dx}{dt} + \frac{dy}{dt} = rate\ of\ change\ of\ tip\ of\ shadow$

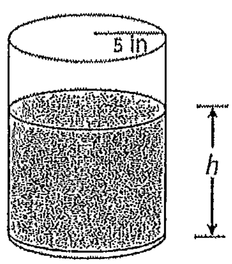
2. A street light is mounted at the top of a 15 ft pole. A man 5 ft tall walks towards the pole at a rate of 5ft per second. A) How fast is the tip of his shadow moving when he is 40 ft from the pole? B) How fast is the length of the shadow changing when he is 40 ft from the pole?

7

3. A conical tank (vertex down) is 40 feet across the top and 40 feet deep. If water is leaking out of the tank at a rate of 80 cubic feet per minute, find the rate of change of the radius of the water when the water is 8 feet deep. ($V = \frac{1}{3}\pi r^2 h$)



4. **2003 AB problem #5**



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time, t, measured in seconds. The volume, V, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius r and height h is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of h. (This means your answer will contain the variable h)

8

AP Calculus AB   Related Rates WS #3   Name_____________________ Date________

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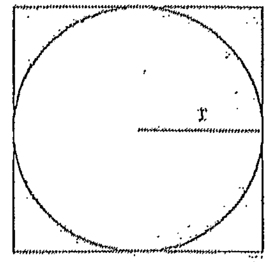
1) A circle is inscribed in a square as shown in the figure. The circumference of the circle is increasing at a constant rate of 6 inches per minute. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)

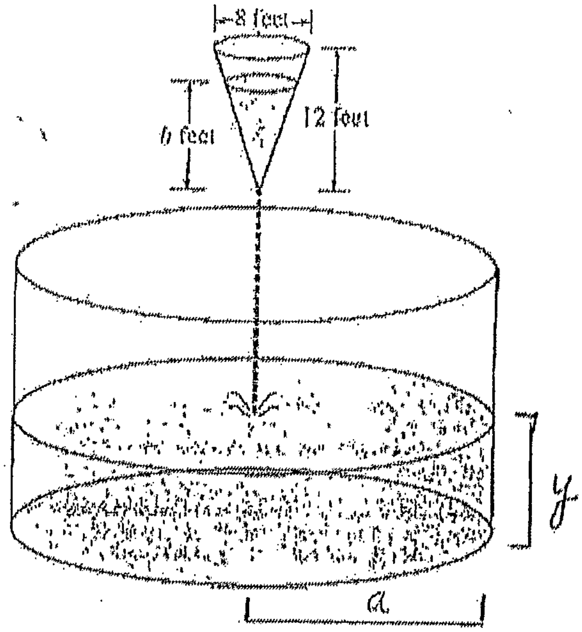


a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

2. Suppose that a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ in^3. How fast is the radius changing after 64 seconds? ($V = \frac{4}{3}\pi r^3$)

3. 1995 AB 5



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V=\frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h=3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h=3$? Indicate units of measure.

$V=\pi a^2 y$

Ch. 4.3 Related Rates Exercise Problems (Day 1)

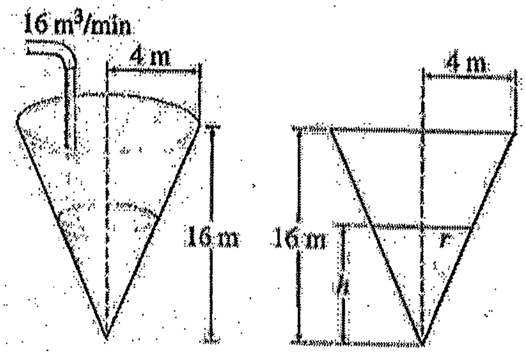
Pg. 286-291 #9, 10, 22, 23, 35, 38

9. Volume of a Cube If each edge of a cube is increasing at the constant rate of 3 cm/s, how fast is the volume of the cube increasing when the length x of an edge is 10 cm?

10. Volume of a Sphere If the radius of a sphere is increasing at 1 cm/s, find the rate of change of its volume when the radius is 6 cm.

22. Filling a Tank Water is flowing into a vertical cylindrical tank of diameter 6 m at the rate of 5 m^3/min. Find the rate at which the depth of the water is rising.

23. Fill Rate A container in the form of a right circular cone (vertex down) has radius 4 m and height 16 m. See the figure. If water is poured into the container at the constant rate of 16 m^3/min, how fast is the water level rising when the water is 8 m deep? *Hint:* The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.



35. Falling Ladder An 8-m ladder is leaning against a vertical wall. If a person pulls the base of the ladder away from the wall at the rate of 0.5 m/s, how fast is the top of the ladder moving down the wall when the base of the ladder is

(a) 3 m from the wall?

(b) 4 m from the wall?

(c) 6 m from the wall?

12

38. **Tracking a Rocket** When a rocket is launched, it is tracked by a tracking dish on the ground located a distance D from the point of launch. The dish points toward the rocket and adjusts its angle of elevation θ to the horizontal (ground level) as the rocket rises. Suppose a rocket rises vertically at a constant speed of 2.0 m/s, with the tracking dish located 150 m from the launch point. Find the rate of change of the angle θ of elevation of the tracking dish with respect to time t (tracking rate) for each of the following:

(a) Just after launch.

(b) When the rocket is 100 m above the ground.

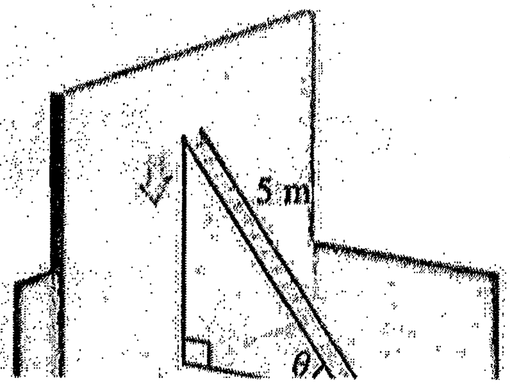
(c) When the rocket is 1.0 km above the ground.

(d) Use the results in (a)–(c) to describe the behavior of the tracking rate as the rocket climbs higher and higher. What limit does the tracking rate approach as the rocket gets extremely high?

Ch. 4.3 Related Rates Exercise Problems (Day 2)

Pg. 286-291 #19, 39, 40, 53, 54

19. Change in Inclination A ladder 5 m long is leaning against a wall. If the lower end of the ladder slides away from the wall at the rate of 0.5 m/s, at what rate is the inclination θ of the ladder with respect to the ground changing when the



39. Lengthening Shadow A child, 1 m tall, is walking directly under a street lamp that is 6 m above the ground. If the child walks away from the light at the rate of 20 m/min, how fast is the child's shadow lengthening?

14

40. Approaching a Pole A boy is walking toward the base of a pole 20 m high at the rate of 4 km/h. At what rate (in meters per second) is the distance from his feet to the top of the pole changing when he is 5 m from the pole?

53. Change in Volume The height h and width x of an open box with a square base are related to its volume by the formula $V = hx^2$. Discuss how the volume changes

(a) if h decreases with time, but x remains constant.

(b) if both h and x change with time.

54. Rate of Change Let $y = 2e^{\cos x}$. If both x and y vary with time in such a way that y increases at a steady rate of 5 units per second, at what rate is x changing when $x = \frac{\pi}{2}$?

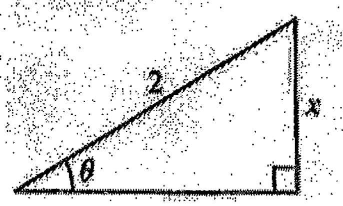
4.3 AP Practice Problems (p. 290-291) – Related Rates

1. A spherical balloon is inflated at the rate of 50 m^3/min. Find the rate at which the radius of the balloon is increasing when the diameter is 20 m.

   (A) $\frac{1}{2\pi}$ m/min (B) $\frac{5}{8\pi}$ m/min

   (C) $\frac{1}{8\pi}$ m/min (D) $\frac{5}{4\pi}$ m/min

2. In the right triangle below, θ is changing at the rate of 2 radians per second. At what rate is x changing at the instant when $x = 1$ cm?



   (A) 2 cm/s (B) $2\sqrt{3}$ cm/s (C) $\sqrt{3}$ cm/s (D) $4\sqrt{3}$ cm/s

3. The radius of a circle is decreasing at a constant rate of 2 in./min. What is the rate of change in the area of the circle when its area is 25π in.^{2}?

   (A) -20π in.^{2}/min (B) -25π in.^{2}/min

   (C) 20π in.^{2}/min (D) $20\pi^2$ in.^{2}/min

16

4. The radius r of a sphere is increasing at a rate of 2 cm/s. At the instant when $r = 12$ cm, what is the rate of change in the surface area S of the sphere? (The surface area S of a sphere with radius r is $S = 4\pi r^2$.)

(A) 96π cm^2/s (B) 1152π cm^2/s

(C) 576π cm^2/s (D) 192π cm^2/s

5. The sides of the rectangle shown below are increasing so that the rate of change of y with respect to time t is three times the rate of change of x with respect to t. If $\frac{dc}{dt} = 1$, what is the rate of change of x when $x = 6$ and $y = 8$?

(A) 3 (B) $\frac{1}{3}$

(C) 1 (D) $\frac{1}{6}$

6. The area of a circle is increasing at a rate of 48π ft^2/h. How fast is the radius of the circle increasing when its area is 36π ft^2?

(A) 4 ft/h (B) 6 ft/h (C) $4\sqrt{3}$ ft/h (D) $\frac{4}{\pi}$ ft/h

7. The radius r and height h of a right circular cone are both increasing at a constant rate of 2 cm/h. At what rate in centimeters cubed per hour is the volume V of the cone increasing when $r = 6$ cm and $h = 15$ cm? (The volume V of a right circular cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)

(A) 24π cm^3/h (B) 96π cm^3/h

(C) 144π cm^3/h (D) 180π cm^3/h

8. Two roads cross at right angles. A police officer sits in a car 65 m east of the crossing and observes a car speeding northbound at 84 m/s. At what speed (in meters per second) is the car distancing itself from the police officer 5 seconds after it passes the crossing?

(A) 166.024 m/s (B) 83.012 m/s

(C) 84 m/s (D) 95.859 m/s

18)

9. A roofer's 13-meter ladder is placed against the wall of a building with its base on level ground. The top of the ladder slips down the wall as the bottom of the ladder slips away from the building at a constant rate of 5 m/s.

(a) At what rate is the top of the ladder moving when it is 5 meters from the ground?

(b) At what rate is the area of the triangle formed by the ladder, the wall, and the ground changing when the top of the ladder is 5 m from the ground?

(c) If θ is the angle formed by the ladder and the ground, what is the rate of change in θ when the top of the ladder is 5 m from the ground?

AP Calculus – 4.4 Notes - L'Hopital's Rule and Indeterminate Form

(19)

Recall: When evaluating limits, first try direct substitution! $\lim_{x \to 3} \frac{2x-5}{x} =$

1. $\lim_{x \to 2} \frac{x^2-7x+10}{x-2} =$

L'Hospital's Rule:

Suppose $f(a) = 0$ and $g(a) = 0$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hopital's Rule allows you to apply the following:

Evaluate each limit. Use L'Hospital's when possible.

2. $\lim_{x \to 2} \frac{x-2}{3x^3-6x^2+x-2}$

3. $\lim_{x \to 0} \frac{\sin(6x)}{x}$

4. $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$

5. $\lim_{x \to \infty} \frac{2x^2}{e^{2x}}$

L'HOSPITAL'S IS NOT THE QUOTIENT RULE!!

6. $\frac{d}{dx} \frac{\sin(6x)}{x}$

(20)

Practice Problems:

Find the following. Use L'Hôpital's when possible.

1. $\lim_{x\to 1} \frac{x-1}{x^2-3x+2}$	2. $\lim_{x\to -5} \frac{x^2-2x-35}{x+5}$	3. $\lim_{x\to 0} \frac{4x}{\ln(x+1)}$
4. $\lim_{x\to 0} \frac{x-1}{x^2-3x+2}$	5. $\lim_{x\to 1} \frac{2(x^2-1)}{\ln x^2}$	6. $\frac{d}{dx} \frac{6x^2+x}{\sin(x)}$

16. If $f(x) = 2x^3 + 5$, then $\lim_{x\to 0} \frac{f(x)-f(0)}{x^3}$ is

(A) 0 (B) 1 (C) 2 (D) 3 (E) The limit does not exist.

17. Functions $f, g,$ and h are twice-differentiable functions with $g(3) = h(3) = 5$. The line $y = 5 + \frac{1}{2}(x-3)$ is tangent to both the graph of g at $x = 3$ and the graph of h at $x = 3$.

a. Find $h'(3)$.

b. Let a be the function given by $a(x) = 2x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(3)$.

c. The function h satisfies $h(x) = \frac{x^2-9}{1-(f(x))^3}$ for $x \neq 3$. It is known that $\lim_{x\to 3} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 3} h(x) = 5$ to find $f(3)$ and $f'(3)$. Show the work that leads to your answers.

EXAMPLE 10 Finding the Limit of an Indeterminate Form of the Type 1^∞

Find $\lim_{x\to 0^+}(1+x)^{1/x}$.

Solution

The expression $(1+x)^{1/x}$ is an indeterminate form at 0^+ of the type 1^∞.

Step 1 Let $y=(1+x)^{1/x}$. Then $\ln y = \frac{1}{x}\ln(1+x)$.

Step 2 $$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \underset{\uparrow}{=} \lim_{x\to 0^+}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1} = 1$$

Type $\frac{0}{0}$; use L'Hôpital's Rule

Step 3 Since $\lim_{x\to 0^+}\ln y = 1$, $\lim_{x\to 0^+} y = e^1 = e$. ■

NOW WORK Problem 85.

4.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$ if $\lim_{x\to c}\frac{f(x)}{g(x)}$ does not exist.
2. *True or False* If $\frac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\frac{0}{0}$, then L'Hôpital's Rule states that $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\left[\frac{d}{dx}\frac{f(x)}{g(x)}\right]$.
3. *True or False* $\frac{1}{x}$ is an indeterminate form at 0.
4. *True or False* $x\ln x$ is not an indeterminate form at 0^+ because $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+}\ln x = -\infty$, and $0\cdot -\infty = 0$.
5. In your own words, explain why $\infty - \infty$ is an indeterminate form, but $\infty + \infty$ is not an indeterminate form.
6. In your own words, explain why $0\cdot\infty \neq 0$.

Skill Building

In Problems 7–26:
(a) Determine whether each expression is an indeterminate form at c.
(b) If it is, identify the type. If it is not an indeterminate form, state why.

7. $\frac{1-e^x}{x}$, $c=0$
8. $\frac{1-e^x}{x-1}$, $c=0$
9. $\frac{e^x}{x}$, $c=0$
10. $\frac{e^x}{x}$, $c=\infty$
11. $\frac{\ln x}{x^2}$, $c=\infty$
12. $\frac{\ln(x+1)}{e^x-1}$, $c=0$
13. $\frac{\sec x}{x}$, $c=0$
14. $\frac{x}{\sec x - 1}$, $c=0$
15. $\frac{\sin x(1-\cos x)}{x^2}$, $c=0$
16. $\frac{\sin x - 1}{\cos x}$, $c=\frac{\pi}{2}$
17. $\frac{\tan x - 1}{\sin(4x-\pi)}$, $c=\frac{\pi}{4}$
18. $\frac{e^x - e^{-x}}{1-\cos x}$, $c=0$
19. x^2e^{-x}, $c=\infty$
20. $x\cot x$, $c=0$
21. $\csc\frac{x}{2} - \cot\frac{x}{2}$, $c=0$
22. $\frac{x}{x-1}+\frac{1}{\ln x}$, $c=1$
23. $\left(\frac{1}{x^2}\right)^{\sin x}$, $c=0$
24. $(e^x+x)^{1/x}$, $c=0$
25. $(x^2-1)^x$, $c=0$
26. $(\sin x)^x$, $c=0$

In Problems 27–42, identify each quotient as an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then find the limit.

27. $\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}$
28. $\lim_{x\to 1}\frac{2x^3+5x^2-4x-3}{x^3+x^2-10x+8}$
29. $\lim_{x\to 1}\frac{\ln x}{x^2-1}$
30. $\lim_{x\to 0}\frac{\ln(1-x)}{e^x-1}$
31. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$
32. $\lim_{x\to 0}\frac{\tan(2x)}{\ln(1+x)}$
33. $\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}$
34. $\lim_{x\to\pi}\frac{1+\cos x}{\sin(2x)}$
35. $\lim_{x\to\infty}\frac{x^2}{e^x}$
36. $\lim_{x\to\infty}\frac{e^x}{x^4}$
37. $\lim_{x\to\infty}\frac{\ln x}{e^x}$
38. $\lim_{x\to\infty}\frac{x+\ln x}{x\ln x}$
39. $\lim_{x\to 0}\frac{e^x-1-\sin x}{1-\cos x}$
40. $\lim_{x\to 0}\frac{e^x-e^{-x}-2\sin x}{3x^3}$
41. $\lim_{x\to 0}\frac{\sin x - x}{x^3}$
42. $\lim_{x\to 0}\frac{x^3}{\cos x - 1}$

In Problems 43–58, identify each expression as an indeterminate form of the type $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, or ∞^0. Then find the limit.

43. $\lim_{x \to 0^+} (x^2 \ln x)$

44. $\lim_{x \to \infty} (xe^{-x})$

PAGE 297 45. $\lim_{x \to \infty} [x(e^{1/x} - 1)]$

46. $\lim_{x \to \pi/2} [(1 - \sin x) \tan x]$

PAGE 298 47. $\lim_{x \to \pi/2} (\sec x - \tan x)$

48. $\lim_{x \to 0} \left(\cot x - \frac{1}{x} \right)$

49. $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{x}{\ln x} \right)$

50. $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

PAGE 298 51. $\lim_{x \to 0^+} (2x)^{3x}$

52. $\lim_{x \to 0^+} x^{x^2}$

53. $\lim_{x \to \infty} (x + 1)^{e^{-x}}$

54. $\lim_{x \to \infty} (1 + x^2)^{1/x}$

55. $\lim_{x \to 0^+} (\csc x)^{\sin x}$

56. $\lim_{x \to \infty} x^{1/x}$

57. $\lim_{x \to \pi/2^-} (\sin x)^{\tan x}$

58. $\lim_{x \to 0} (\cos x)^{1/x}$

In Problems 59–88, find each limit.

59. $\lim_{x \to 0^+} \frac{\cot x}{\cot(2x)}$

60. $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$

PAGE 296 61. $\lim_{x \to 1/2^-} \frac{\ln(1 - 2x)}{\tan(\pi x)}$

62. $\lim_{x \to 1^-} \frac{\ln(1 - x)}{\cot(\pi x)}$

63. $\lim_{x \to \infty} \frac{x^4 + x^3}{e^x + 1}$

64. $\lim_{x \to \infty} \frac{x^2 + x - 1}{e^x + e^{-x}}$

65. $\lim_{x \to 0} \frac{xe^{4x} - x}{1 - \cos(2x)}$

66. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$

67. $\lim_{x \to 0} \frac{\tan^{-1} x}{x}$

68. $\lim_{x \to 0} \frac{\tan^{-1} x}{\sin^{-1} x}$

69. $\lim_{x \to 0} \frac{\cos x - 1}{\cos(2x) - 1}$

70. $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

71. $\lim_{x \to 0^+} (x^{1/2} \ln x)$

72. $\lim_{x \to \infty} [(x - 1)e^{-x^2}]$

73. $\lim_{x \to \pi/2} [\tan x \ln(\sin x)]$

74. $\lim_{x \to 0^+} [\sin x \ln(\sin x)]$

75. $\lim_{x \to 0} [\csc x \ln(x + 1)]$

76. $\lim_{x \to \pi/4} [(1 - \tan x) \sec(2x)]$

77. $\lim_{x \to a} \left[(a^2 - x^2) \tan\left(\frac{\pi x}{2a} \right) \right]$

78. $\lim_{x \to 1^+} \left[(1 - x) \tan\left(\frac{1}{2} \pi x \right) \right]$

79. $\lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x - 1} \right)$

80. $\lim_{x \to 1} \left(\frac{x}{x - 1} - \frac{1}{\ln x} \right)$

81. $\lim_{x \to \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x \right)$

82. $\lim_{x \to \pi} (\cot x - x \csc x)$

83. $\lim_{x \to 1^-} (1 - x)^{\tan(\pi x)}$

84. $\lim_{x \to 0^+} x^{\sqrt{x}}$

PAGE 299 85. $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x}$

86. $\lim_{x \to \infty} \left(1 + \frac{5}{x} + \frac{3}{x^2} \right)^x$

87. $\lim_{x \to (\pi/2)^-} (\tan x)^{\cos x}$

88. $\lim_{x \to 0^+} (x^2 + x)^{-\ln x}$

Applications and Extensions

89. **Wolf Population** In 2014 there were 229 wolves in Wyoming outside of Yellowstone National Park. Suppose the population w of wolves in the region at time t follows the logistic growth curve

$$w = w(t) = \frac{Ke^{rt}}{\frac{K}{40} + e^{rt} - 1}$$

where $K = 252$, $r = 0.283$, and $t = 0$ represents the population in the year 2000.

Source: Federal Wildlife Service.

(a) Find $\lim_{t \to \infty} w(t)$.

(b) Interpret the answer found in (a) in the context of the problem.

(c) Use technology to graph $w = w(t)$.

90. **Skydiving** The downward velocity v of a skydiver with nonlinear air resistance can be modeled by

$$v = v(t) = -A + RA \frac{e^{Bt+C} - 1}{e^{Bt+C} + 1}$$

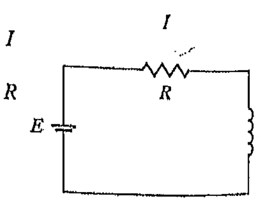
where t is the time in seconds, and A, B, C, and R are positive constants with $R > 1$.

(a) Find $\lim_{t \to \infty} v(t)$.

(b) Interpret the limit found in (a).

(c) If the velocity v is measured in feet per second, reasonable values of the constants are $A = 108.6$, $B = 0.554$, $C = 0.804$, and $R = 2.62$. Graph the velocity of the skydiver with respect to time.

91. **Electricity** The equation governing the amount of current I (in amperes) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is $I = \frac{E}{R}(1 - e^{-Rt/L})$.



(a) Find $\lim_{t \to \infty} I(t)$ and $\lim_{R \to 0^+} I(t)$.

(b) Interpret these limits.

92. Find $\lim_{x \to 0} \frac{a^x - b^x}{x}$, where $a \neq 1$ and $b \neq 1$ are positive real numbers.

93. Show that $\lim_{x \to \infty} \frac{\ln x}{x^n} = 0$, for $n \geq 1$ an integer.

94. Show that $\lim_{x \to \infty} \frac{x^n}{e^x} = 0$ for $n \geq 1$ an integer.

95. Show that $\lim_{x \to 0^+} (\cos x + 2 \sin x)^{\cot x} = e^2$.

96. Find $\lim_{x \to \infty} \frac{P(x)}{e^x}$, where P is a polynomial function.

97. Find $\lim_{x \to \infty} [\ln(x + 1) - \ln(x - 1)]$.

98. Show that $\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x} = 0$. *Hint:* Write $\frac{e^{-1/x^2}}{x} = \frac{\frac{1}{x}}{e^{1/x^2}}$.

99. If n is an integer, show that $\lim_{x \to 0^+} \frac{e^{-1/x^2}}{x^n} = 0$.

100. Show that $\lim_{x \to \infty} \sqrt[x]{x} = 1$.