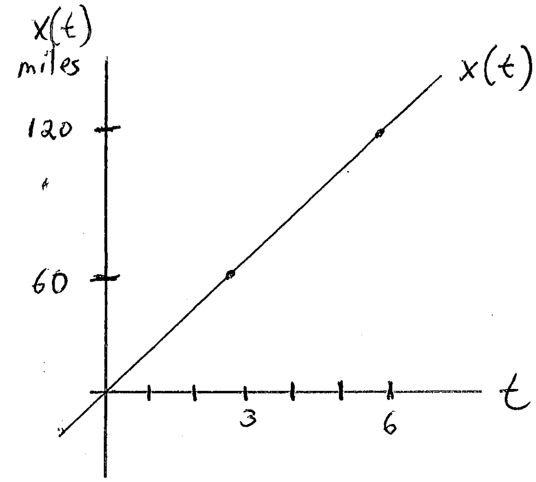
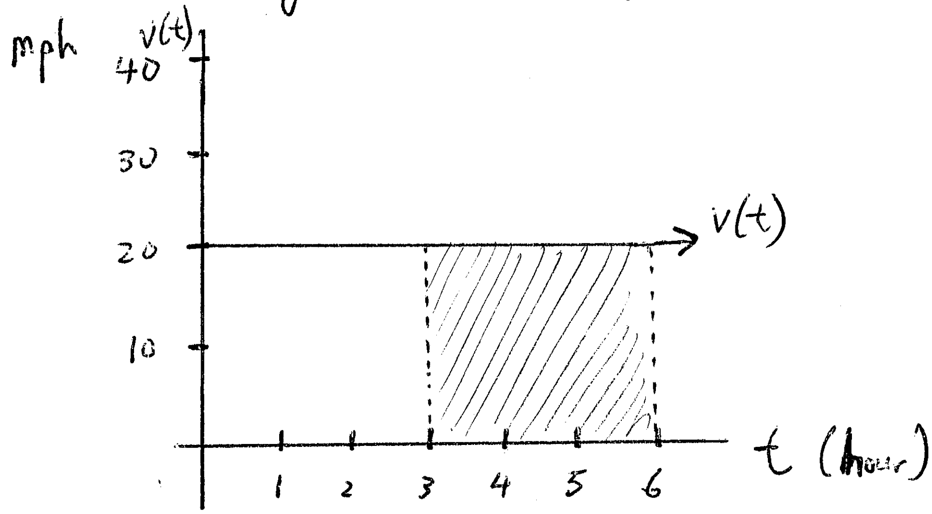
position function $x(t) = 20t$



Velocity function $v(t) = 20$



Displacement : $\int_3^6 v(t)\,dt = 20 \text{ miles}/\text{hr} \cdot 3 \text{ hrs.} = 60 \text{ miles}$

$\int_3^6 v(t) = x(6) - x(3) = 120 - 60 = 60 \text{ miles}$

$x(6) = 20(6) = 120$

$x(3) = 20(3) = 60$

$$A_f(x) = \int_2^x (t^2 + 10)\,dt = \frac{1}{3}x^3 + 10x - 22\tfrac{2}{3}, \text{ and thus}$$

$$A_f(3) = \frac{1}{3} \cdot 3^3 + 10 \cdot 3 - 22\tfrac{2}{3}$$

This is identical to the last line of the previous computation. Thus, finding the area between 2 and 3 by subtracting $F(2)$ from $F(3)$ is mathematically equivalent to computing $A_f(3)$ for the area function that starts at $s = 2$.

Why the theorem works: The integration/differentiation connection

I know, I know. You're asking, "A third explanation?" Okay, maybe I've gone a bit overboard with all these explanations, but don't skip this one — it's the best way to understand the second version of the Fundamental Theorem and why integration is the reverse of differentiation. Take my word for it — it's worth the effort. Consider Figure 14-8.

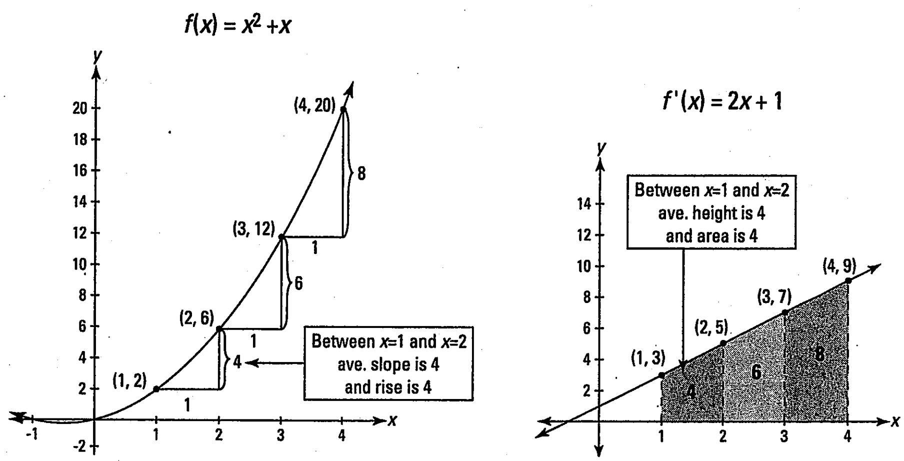


Figure 14-8: The essence of differentiation and integration in a single figure! It's a yin/yang thing.

Figure 14-8 shows a function, $x^2 + x$, and its derivative, $2x + 1$. Look carefully at the numbers 4, 6, and 8 on both graphs. The connection between 4, 6, and 8 on the graph of f — which are the amounts of *rise* between subsequent points on the curve — and 4, 6, and 8 on the graph of f' — which are the *areas* of the trapezoids under f' — shows the intimate relationship between integration and differentiation. Figure 14-8 is probably the single most important figure in this book. It's a picture worth a thousand symbols and equations, encapsulating the essence of integration in a single snapshot. It shows how the second version of the Fundamental Theorem works because it shows that the *area* under $2x + 1$ between 1 and 4 equals the total *rise* on f between (1, 2) and (4, 20), in other words that

$$\int_1^4 f'(x)\,dx = f(4) - f(1)$$

Note that I've called the two functions in Figure 14-8 and in the above equation f and f' to emphasize that $2x + 1$ is the *derivative* of $x^2 + x$. I could have instead called $x^2 + x$ F and called $2x + 1$ f, which would emphasize that $x^2 + x$ is an *antiderivative* of $2x + 1$. In that case you would write the previous area equation in the standard way,

$$\int_1^4 f(x)\,dx = F(4) - F(1)$$

Either way, the meaning's the same. I use the derivative version to point out how finding area is differentiation in reverse. Going from left to right in Figure 14-8 is differentiation: The heights on f' give you the slopes of f. Going from right to left is integration: The change between two heights on f gives you an area under f'.

Okay, here's how it works. Imagine you're going up along f from (1, 2) to (2, 6). Every point along the way has a certain steepness, a slope. This slope is plotted as the y-coordinate, or height, on the graph of f'. The fact that f' goes up from (1, 3) to (2, 5) tells you that the slope of f goes up from 3 to 5 as you travel between (1, 2) and (2, 6). This all follows from basic differentiation.

Now, as you go along f from (1, 2) to (2, 6), the slope is constantly changing. But it turns out that because you go up a total *rise* of 4 as you *run* across 1, the average of all the slopes on f between (1, 2) and (2, 6) is $\frac{4}{1}$, or 4. Because each of these slopes is plotted as a y-coordinate or height on f', it follows that the average height of f' between (1, 3) and (2, 5) is also 4. Thus, between two given points, average slope on f equals average height on f'.

Hold on, you're almost there. *Slope* equals $\frac{rise}{run}$, so when the run is 1, the slope equals the rise. For example, from (1, 2) to (2, 6) on f, the curve rises up 4 and the average slope between those points is also 4. Thus, between two given points on f, the average slope *is* the rise.

The area of a trapezoid like the ones on the right in Figure 14-8 equals its width times its average height. (This is true of any other similar shape that has a bottom like a rectangle; the top can be any crooked line or funky curve you like.) So, because the width of each trapezoid is 1, and because anything times 1 is itself, the average height of each trapezoid under f' *is* its area; for instance, the area of that first trapezoid is 4 and its average height is also 4.

Are you ready for the grand finale? Here's the whole argument in a nutshell. On f, *rise = average slope*; going from f to f', *average slope = average height*; on f', *average height = area*. So that gives you *rise = slope = height = area*, and thus, finally, *rise = area*. And that's what the second version of the Fundamental Theorem says:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\textit{area} = \textit{rise}$$

Piece o' cake, right? (This is just a guess, but in the event that you find this less than crystal clear, I suspect that it's not going to make much difference to you that I'm quite pleased with what I've just written.) All kidding aside, this is unavoidably difficult. You may have to read it two or three times for it to really sink in.

Notice that it makes no difference to the relationship between slope and area if you use any other function of the form $x^2 + x + C$ instead of $x^2 + x$. Any parabola like $x^2 + x + 10$ or $x^2 + x - 5$ is exactly the same shape as $x^2 + x$ — it's just been slid up or down vertically. Any such parabola rises up between $x = 1$ and $x = 4$ in precisely the same way as the parabola in Figure 14-8. From 1 to 2 these parabolas go over 1, up 4. From 2 to 3 they go over 1, up 6, and so on. This is why any antiderivative can be used to find area. The total *area* under f' between 1 to 4, namely 18, corresponds to the total *rise* on any of these parabolas from 1 to 4, namely 4 + 6 + 8, or 18.

Well, there you have it — actual explanations of why the shortcut version of the Fundamental Theorem works and why finding area is differentiation in reverse. If you understand only half of what I've just written, you're way ahead of most students of calculus. The good news is that you probably won't be tested on this theoretical stuff.

Now let's come back down to earth.

Finding Antiderivatives: Three Basic Techniques

I've been doing a lot of talking about antiderivatives, but just how do you find them? In this section, I give you three easy techniques. Then in Chapter 15, I give you four advanced techniques. In case you're curious, you *will* be tested on this stuff.