Key

**AP Calculus AB** **Final Exam Free Response Review Problems**

1) For the function $g(x) = \begin{cases} \frac{x^2+x-2}{x-1}, x \neq 1 \\ 5, x = 1 \end{cases}$

* continuity conditions:
a) $f(c)$ exists
b) $\lim_{x\to c} f(x)$ exist $\left(\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\right)$
c) $f(c) = \lim_{x\to c} f(x)$

a) Use limits to show that $g(x)$ is discontinuous at $x = 1$ and state why it is discontinuous there.
b) Determine if the discontinuity is removable or non-removable and state why.

a)
i) $g(1) = 5$
ii) $\lim_{x\to 1} \frac{x^2+x-2}{x-1} = \lim_{x\to 1} \frac{(x+2)\cancel{(x-1)}}{\cancel{(x-1)}} = 1+2 = 3$
iii) $g(1) \neq \lim_{x\to 1} g(x)$

Since $g(1) \neq \lim_{x\to 1} g(x)$, $g(x)$ is discontinuous at $x=1$.

b) $g(x)$ has removable discontinuity at $x=1$ since $\lim_{x\to 1} g(x)$ exists but $g(1) \neq \lim_{x\to 1} g(x)$

2) Find the point on $y = \sqrt{3x-1}$ where the tangent line is perpendicular to the line $3y + 2x = 3$. Then write an equation of the tangent line to the curve $y = \sqrt{3x-1}$ at that point

$3y + 2x = 3$
$3y = -2x + 3$
$y = \frac{-2}{3}x + 1$
slope is $m_1 = -2/3$
$\perp$ slope is $m_2 = 3/2$

* find where the curve $y = \sqrt{3x-1}$ has slope of $m = 3/2$
* set $y'(x) = 3/2$

$y = (3x-1)^{1/2}$
$y' = \frac{1}{2}(3x-1)^{-1/2}(3)$
$y' = \frac{3}{2\sqrt{3x-1}}$

$\frac{3}{2\sqrt{3x-1}} = \frac{3}{2}$
$6\sqrt{3x-1} = 6$
$\sqrt{3x-1} = 1$

$\left(\sqrt{3x-1}\right)^2 = (1)^2$
$3x - 1 = 1$
$3x = 2$
$x = 2/3$
$y(2/3) = \sqrt{3(2/3)-1} = 1$

point: $(2/3, 1)$
slope: $m = 3/2$
$y - y_1 = m(x - x_1)$
$\boxed{y - 1 = \frac{3}{2}\left(x - \frac{2}{3}\right)}$

3) Liquid is being poured into a large vat. After $t$ hours, the amount of gallons of liquid in the vat can be represented by $V(t) = 5t - \sqrt{t}$.

a) What is the average rate of liquid poured into the vat over the first 4 hours ($t = 0$ to $t = 4$)? (Include units of measure)

slope between endpoints

b) At what rate is the liquid being poured into the vat when $t = 4$? (Include units of measure)

a) $v(0) = 5(0) - \sqrt{0} = 0$
$v(4) = 5(4) - \sqrt{4} = 18$

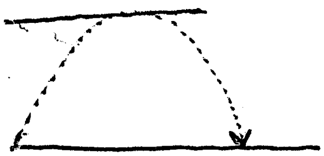
Avg. Rate of Liquid $= \frac{v(4) - v(0)}{4 - 0} = \frac{18 - 0}{4 - 0} = \boxed{9/2 \text{ gallons/hr.}}$

b) * to find instantaneous rate, find $v'(4)$

$v'(t) = 5 - \frac{1}{2}t^{-1/2} = 5 - \frac{1}{2\sqrt{t}}$

$v'(4) = 5 - \frac{1}{2\sqrt{4}} = 5 - \frac{1}{2(2)} = 5 - 1/4 = \boxed{\frac{19}{4} \text{ gallons/hr.}}$

4) A ball is thrown straight up in the air from a point 64 feet above ground level so that its position function is $h(t) = -16t^2 + 48t + 64$, where $t$ is measured in seconds. Use this to answer the following questions. Include units with all answers.



$h(t) = -16t^2 + 48t + 64$

$v(t) = -32t + 48$

$a(t) = -32$

a) What is the height of the ball at $t = 2$?

$h(2) = -16(2)^2 + 48(2) + 64$

$= -64 + 96 + 64$

$= \boxed{96 \text{ feet}}$

b) What is the velocity at $t = 2$?

$v(2) = -32(2) + 48$

$= -64 + 48$

$= \boxed{-16 \text{ ft/s}}$

c) When does the ball reach its greatest height?

* ball reaches highest point when $v(t) = 0$

* set $v(t) = 0$

$-32t + 48 = 0$

$-32t = -48$

$t = \frac{-48}{-32} = \frac{3}{2}$

$\boxed{t = 3/2 \text{ sec}}$

d) What is the greatest height?

$h(3/2) = -16(9/4) + 48(3/2) + 64$

$= -36 + 72 + 64 = \boxed{100 \text{ ft}}$

e) At what time is the ball **falling** at a speed of 48 feet per second?

* set $v(t) = -48$

$-32t + 48 = -48$

$-32t = -96$

$\boxed{t = 3 \text{ sec}}$

f) At what time does the ball hit the ground? ball hits ground when $h(t) = 0$

$-16t^2 + 48t + 64 = 0$

$-16(t^2 - 3t - 4) = 0$

$-16(t-4)(t+1) = 0$

$t = 4, -1$

$\boxed{t = 4 \text{ sec}}$

g) With what velocity does the ball hit the ground?

* ball hits ground at $t = 4$, find $v(4)$:

$v(4) = -32(4) + 48 = \boxed{-80 \text{ ft/sec}}$

h) What is the ball's average velocity from $t = 0$ to $t = 2$?

$\text{Avg. velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{h(2) - h(0)}{2 - 0}$

$= \frac{96 - 64}{2 - 0} = \boxed{16 \text{ ft/sec}}$

i) What is the ball's acceleration at $t = 2$?

$a(2) = \boxed{-32 \text{ ft/sec}^2}$

j) Is the speed increasing or decreasing at $t = 2$?

$v(2) = -16 \text{ ft/s}, \; a(2) = -32 \text{ ft/s}^2$

Since $v(2)$ and $a(2)$ have same signs, speed is increasing at $t = 2$ sec.

5) Let $f$ be the function defined by $f(x) = xe^{(1-x)}$ for all real numbers $x$.

$\lim_{x\to\infty} xe^{1-x} = \frac{x}{e^{x-1}} = 0$

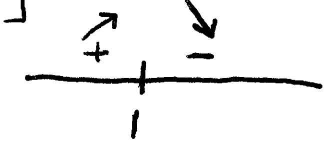
a) Find each interval on which $f$ is increasing. Justify your answer.
b) Find the range of $f$.
c) Find the each point of inflection of the graph of $f$. Justify your answer.
d) Using the results found in parts a, b, and c, sketch the graph of $f$.

$\lim_{x\to-\infty} xe^{1-x} = -\infty$

product rule

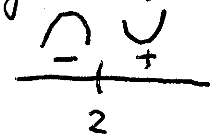
a) $f'(x) = 1e^{(1-x)} + x\cdot e^{(1-x)}(-1) = e^{1-x}[1-x]$

set $f'(x)=0$, $e^{1-x}(1-x)=0$, $\underline{x=1}$




$f(x)$ increasing $(-\infty, 1)$ b/c $f'(x) > 0$

b) Max at $f(1) = 1e^0 = 1$ so Range is $(-\infty, 1]$



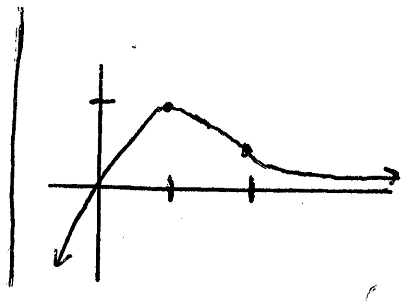

product Rule

c) $f''(x) = e^{1-x}(-1)[1-x] + e^{1-x}(-1)$

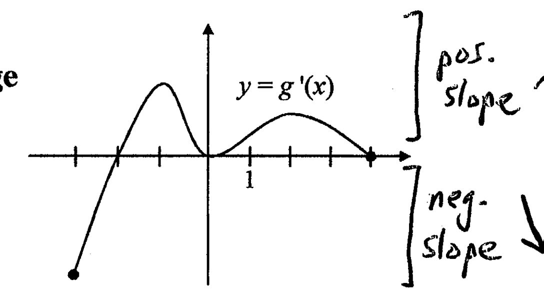
$= e^{1-x}[-1+x-1] = \underline{e^{1-x}(x-2)}$

POI at $(2, \frac{2}{e})$

set $f''(x)=0$ $e^{1-x}(x-2)=0$, $\underline{x=2}$

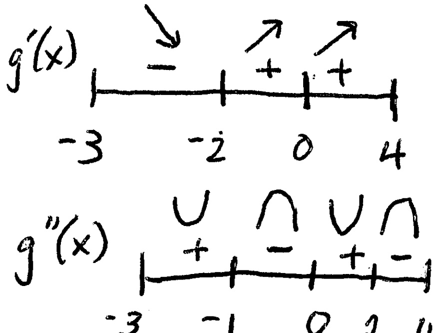


6) To the right is the graph of $g'(x)$, the **derivative** of a continuous function, $g$. The domain of $g$ is $[-3, 4]$, **the range of $g$ is $[-3, 2]$**, and $g(-3) = -2$, $g(0) = 0$, and $g(2) = 1$.




Find the following. Justify your answers.

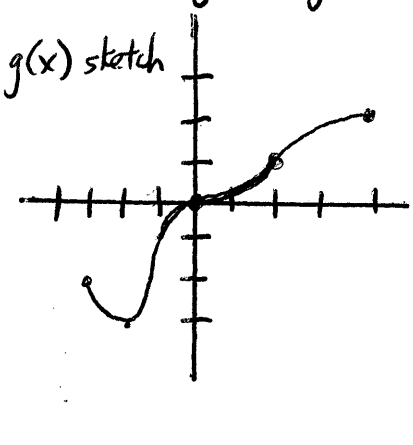
a) interval(s) where $g$ is increasing
b) interval(s) where $g$ is concave down
c) ***x*-coordinate** of each rel. min
d) ***x*-coordinate** of each pt. of inflection
e) Sketch the graph of the function $y = g(x)$




c) Rel. min at $x = -2$ b/c $g'(x)$ changes from $-$ to $+$

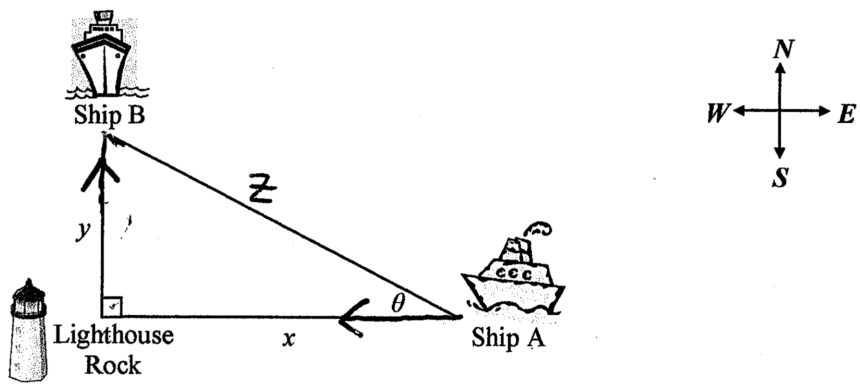
d) POI at $x = -1, 0, 2$ b/c $g''(x)$ change signs

e) $g(x)$ sketch



a) $g(x)$ is increasing on $(-2, 0) \cup (0, 4)$ b/c $g'(x) > 0$

b) $g(x)$ concave down $(-1, 0) \cup (2, 4)$ b/c $g''(x) < 0$




7) Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour. Ship B is traveling due north away from Lighthouse Rock at a speed of 10 kilometers per hour. Let $x$ be the distance between Ship A and Lighthouse Rock at time $t$, and let $y$ be the distance between Ship B and Lighthouse Rock at time $t$, as shown in the figure above.

a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.
b) Find the rate of change, in kilometers per hour, of the distance between the two ships when $x = 4$ km and $y = 3$ km.
c) Let $\theta$ be the angle shown in the figure. Find the rate of change of $\theta$, in radians per hour, when $x = 4$ km and $y = 3$ km.

$\frac{dx}{dt} = -15$ km/hr $\quad \frac{dy}{dt} = +10$ Km/hr

a) $x^2 + y^2 = z^2, \quad 3^2 + 4^2 = z^2$

$\boxed{z = 5\text{ Km}}$

b) Find $\frac{dz}{dt}$: $x^2 + y^2 = z^2$

$x = 4 \quad \frac{dx}{dt} = -15$

$y = 3 \quad \frac{dy}{dt} = +10$

$z = 5 \quad \frac{dz}{dt} = \_\_\_$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$2(4)(-15) + 2(3)(+10) = 2(5)\left(\frac{dz}{dt}\right)$

$-120 + 60 = 10\left(\frac{dz}{dt}\right)$

$-60 = 10\frac{dz}{dt}$

$\boxed{\frac{dz}{dt} = -6\text{ Km/hr.}}$

c) $\tan\theta = \frac{y}{x}$ $\quad \sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{\left(\frac{dy}{dt}\right)(x) - y\left(\frac{dx}{dt}\right)}{x^2}$

$\sec\theta = \frac{5}{4}$

$\left(\frac{5}{4}\right)^2\left(\frac{d\theta}{dt}\right) = \frac{(10)(4) - 3(-15)}{16}$

$\frac{25}{16}\left(\frac{d\theta}{dt}\right) = \frac{40+45}{16}$

$\frac{d\theta}{dt} = \frac{16}{25} \cdot \frac{85}{16} = \frac{85}{25} = \boxed{\frac{17}{5}\text{ rad/hr}}$