Name: ______________________________ Period: __________

AB Calculus

Unit 3

Differentiation Part 2

(Composite, Implicit, & Derivative Review)

Ch. 3.1 Notes: The Chain Rule

Chain Rule: Method of computing the derivative of the composition of 2 or more functions (function within a function)

* Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Steps:

1) Take the derivative of the outside while keeping the inside portion unchanged
2) Then multiply by the derivative of the inside function.

$\boxed{\text{Ex. 1}}$ $f(x) = (3x^2+2)^5$

$f'(x) =$

$\boxed{\text{Ex. 2}}$ Find all values of x of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and where $f'(x)$ does not exist

(2)

$\boxed{\text{Ex. 3}}$ $y = \frac{4}{(x+2)^2}$ find equation of tangent line to y at $x=-3$

$\boxed{\text{Ex. 4}}$ $y = \left(\frac{x-1}{x^2-4}\right)^3$

$\boxed{\text{Ex. 5}}$ $y = \frac{x}{\sqrt{x^2-1}}$

3.1 Chain Rule Practice Problems WS #1

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Finding a Derivative **In Exercises 7–34, find the derivative of the function.**

Chain Rule: $\frac{d}{dx}[f(g(x)] = f'[g(x)] * g'(x)$

1. $y = (5x - 8)^4$

2) $y = (4x - 1)^3$

3) $y = 5(2 - x^3)^4$

4) $g(x) = 3(4 - 9x)^4$

5) $f(t) = \sqrt{5 - t}$

6) $y = \sqrt[3]{6x^2 + 1}$

7) $f(x) = \sqrt{x^2 - 4x + 2}$

8) $y = 2\sqrt[4]{9 - x^2}$

4

Find the derivative of the function below:

Chain Rule: $\frac{d}{dx}[f(g(x)] = f'[g(x)] * g'(x)$

9) $y = \dfrac{1}{x-2}$

10) $y = \dfrac{1}{\sqrt{3x+5}}$

11) $y = \dfrac{x}{\sqrt{x^2+1}}$

12) $y = \dfrac{x}{\sqrt{x^4+4}}$

13) $g(x) = \left(\dfrac{x+5}{x^2+2}\right)^2$

14) $g(x) = \left(\dfrac{3x^2-2}{2x+3}\right)^3$

3.1 Chain Rule Practice Problems WS #1

Key

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Finding a Derivative In Exercises 7–34, find the derivative of the function.

Chain Rule: $\frac{d}{dx}[f(g(x)] = f'[g(x)] * g'(x)$

1. $y = (5x - 8)^4$

outside: $(\ )^4$
inside: $5x-8$

$y' = 4(\ \ )^3 \cdot (5)$

$y' = 4(5x-8)^3 \cdot 5$

$\boxed{y' = 20(5x-8)^3}$

2) $y = (4x - 1)^3$

outside: $(\ )^3$
inside: $4x-1$

$y' = 3(4x-1)^2 \cdot (4)$

$\boxed{y' = 12(4x-1)^2}$

3) $y = 5(2 - x^3)^4$

outside: $5(\ )^4$
inside: $2-x^3$

$y' = 5 \cdot 4(\ \ )^3 \cdot (-3x^2)$

$y' = 20(2-x^3)^3 \cdot -3x^2$

$\boxed{y' = -60x^2(2-x^3)^3}$

4) $g(x) = 3(4 - 9x)^4$

outside: $3(\ )^4$
inside: $4-9x$

$g'(x) = 3 \cdot 4(\ \ )^3 \cdot (-9)$

$g'(x) = 12(4-9x)^3(-9)$

$\boxed{g'(x) = -108(4-9x)^3}$

5) $f(t) = \sqrt{5 - t}$

outside: $(\ )^{1/2}$
inside: $5-t$

$f(t) = (5-t)^{1/2}$

$f'(t) = \frac{1}{2}(\ \ )^{-1/2}(-1)$

$f'(t) = \frac{1}{2}(5-t)^{-1/2}(-1)$

$\boxed{f'(t) = \frac{-1}{2(5-t)^{1/2}}}$

6) $y = \sqrt[3]{6x^2 + 1}$

outside: $(\ \ )^{1/3}$
inside: $6x^2+1$

$y = (6x^2+1)^{1/3}$

$y' = \frac{1}{3}(\ \ )^{-2/3}(12x)$

$y' = \frac{1}{3}(6x^2+1)^{-2/3} \cdot 12x$

$\boxed{y' = \frac{4x}{(6x^2+1)^{2/3}}}$

7) $f(x) = \sqrt{x^2 - 4x + 2}$

outside: $(\ )^{1/2}$
inside: x^2-4x+2

$f(x) = (x^2-4x+2)^{1/2}$

$f'(x) = \frac{1}{2}(\ \ )^{-1/2} \cdot (2x-4)$

$f'(x) = \frac{1}{\cancel{2}}(x^2-4x+2)^{-1/2} \cdot \cancel{2}(x-2)$

$\boxed{f'(x) = \frac{x-2}{(x^2-4x+2)^{1/2}}}$

8) $y = 2\sqrt[4]{9 - x^2}$

outside: $2(\ )^{1/4}$
inside: $9-x^2$

$y = 2(9-x^2)^{1/4}$

$y' = 2 \cdot \frac{1}{4}(\ \ )^{-3/4} \cdot (-2x)$

$y' = \frac{2}{4}(9-x^2)^{-3/4}(-2x)$

$\boxed{y' = \frac{-x}{(9-x^2)^{3/4}}}$

6

Find the derivative of the function below:

Chain Rule: $\frac{d}{dx}[f(g(x)] = f'[g(x)] * g'(x)$

9) $y = \frac{1}{x-2}$

outside: $(\ )^{-1}$
inside: $x-2$

$y=(x-2)^{-1}$

$y'=-1(x-2)^{-2}(1)$

$\boxed{y'=\frac{-1}{(x-2)^2}}$

10) $y = \frac{1}{\sqrt{3x+5}}$

outside: $(\ )^{-1/2}$
inside: $3x+5$

$y=\frac{1}{(3x+5)^{1/2}}$

$y=(3x+5)^{-1/2}$

$y'=\frac{-1}{2}(\ \ )^{-3/2}(3)$

$y'=\frac{-1}{2}(3x+5)^{-3/2}(3)$

$\boxed{y'=\frac{-3}{2(3x+5)^{3/2}}}$

11) $y = \frac{x}{\sqrt{x^2+1}}$

1) quotient
2) chain
outside: $(\ )^{1/2}$
inside: x^2+1

$y=\frac{x}{(x^2+1)^{1/2}}$

$y'=\frac{(1)(x^2+1)^{1/2}-x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left[(x^2+1)^{1/2}\right]^2}$

$y'=\frac{(x^2+1)^{1/2}-\frac{x^2}{(x^2+1)^{1/2}}}{x^2+1}\cdot\frac{(x^2+1)^{1/2}}{(x^2+1)^{1/2}}$

$y'=\frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}}$

$\boxed{y'=\frac{1}{(x^2+1)^{3/2}}}$

12) $y = \frac{x}{\sqrt{x^4+4}}$

1) quotient
2) chain:
outside: $(\ )^{1/2}$
inside: x^4+4

$y=\frac{x}{(x^4+4)^{1/2}}$

$y'=\frac{(1)(x^4+4)^{1/2}-(x)\cdot\frac{1}{2}(x^4+4)^{-1/2}(4x^3)}{\left[(x^4+4)^{1/2}\right]^2}$

$y'=\frac{(x^4+4)^{1/2}-\frac{2x^4}{(x^4+4)^{1/2}}}{x^4+4}\cdot(x^4+4)^{1/2}$

$y'=\frac{x^4+4-2x^4}{(x^4+4)^{3/2}}$

$\boxed{y'=\frac{4-x^4}{(x^4+4)^{3/2}}}$

13) $g(x) = \left(\frac{x+5}{x^2+2}\right)^2$

1) chain
outside: $(\ )^2$
inside: $\frac{x+5}{x^2+2}$
2) quotient

$g'(x)=2\left[\frac{x+5}{x^2+2}\right]^1\left[\frac{(1)(x^2+2)-(x+5)(2x)}{(x^2+2)^2}\right]$

$g'(x)=\frac{2(x+5)(x^2+2-2x^2-10x)}{(x^2+2)^3}$

$\boxed{g'(x)=\frac{2(x+5)(-x^2-10x+2)}{(x^2+2)^3}}$

14) $g(x) = \left(\frac{3x^2-2}{2x+3}\right)^3$

1) chain
outside: $(\ )^3$
inside: $\frac{3x^2-2}{2x+3}$
2) quotient

$g'(x)=3\left[\frac{3x^2-2}{2x+3}\right]^2\left[\frac{(6x)(2x+3)-(3x^2-2)(2)}{(2x+3)^2}\right]$

$g'(x)=\frac{3(3x^2-2)^2(12x^2+18x-6x^2+4)}{(2x+3)^2(2x+3)^2}$

$\boxed{g'(x)=\frac{3(3x^2-2)^2(6x^2+18x+4)}{(2x+3)^4}}$

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3.1 Assess Your Understanding

Concepts and Vocabulary

1. The derivative of a composite function $(f \circ g)(x)$ can be found using the ______ Rule.
2. *True or False* If $y = f(u)$ and $u = g(x)$ are differentiable functions, then $y = f(g(x))$ is differentiable.
3. *True or False* If $y = f(g(x))$ is a differentiable function, then $y' = f'(g'(x))$.
4. To find the derivative of $y = \tan(1 + \cos x)$, using the Chain Rule, begin with $y =$ ______ and $u =$ ______.
5. If $y = (x^3 + 4x + 1)^{100}$, then $y' =$ ______.
6. If $f(x) = e^{3x^2+5}$, then $f'(x) =$ ______.
7. *True or False* The Chain Rule can be applied to multiple composite functions.
8. $\frac{d}{dx} \sin x^2 =$ ______.

$3x+2$

Skill Building

In Problems 9–14, write y as a function of x. Find $\frac{dy}{dx}$ using the Chain Rule.

9. $y = u^5$, $u = x^3 + 1$
10. $y = u^3$, $u = 2x + 5$
11. $y = \frac{u}{u+1}$, $u = x^2 + 1$
12. $y = \frac{u-1}{u}$, $u = x^2 - 1$
13. $y = (u+1)^2$, $u = \frac{1}{x}$
14. $y = (u^2 - 1)^3$, $u = \frac{1}{x+2}$

$\frac{\pi}{2}x$

In Problems 15–32, find the derivative of each function using the Power Rule for Functions.

15. $f(x) = (3x+5)^2$
16. $f(x) = (2x-5)^3$
17. $f(x) = (6x-5)^{-3}$
18. $f(t) = (4t+1)^{-2}$
19. $g(x) = (x^2+5)^4$
20. $F(x) = (x^3-2)^5$
21. $f(u) = \left(u - \frac{1}{u}\right)^3$
22. $f(x) = \left(x + \frac{1}{x}\right)^3$
23. $g(x) = (4x + e^x)^3$
24. $F(x) = (e^x - x^2)^2$
25. $f(x) = \tan^2 x$
26. $f(x) = \sec^3 x$
27. $f(z) = (\tan z + \cos z)^2$
28. $f(z) = (e^z + 2\sin z)^3$
29. $y = (x^2+4)^2(2x^3-1)^3$
30. $y = (x^2-2)^3(3x^4+1)^2$
31. $y = \left(\frac{\sin x}{x}\right)^2$
32. $y = \left(\frac{x + \cos x}{x}\right)^5$

In Problems 33–54, find y'.

33. $y = \sin(4x)$
34. $y = \cos(5x)$
35. $y = 2\sin(x^2 + 2x - 1)$
36. $y = \frac{1}{2}\cos(x^3 - 2x + 5)$
37. $y = \sin\frac{1}{x}$
38. $y = \sin\frac{3}{x}$
39. $y = \sec(4x)$
40. $y = \cot(5x)$
41. $y = e^{1/x}$
42. $y = e^{1/x^2}$
43. $y = \frac{1}{x^4 - 2x + 1}$
44. $y = \frac{3}{x^5 + 2x^2 - 3}$
45. $y = \frac{100}{1 + 99e^{-x}}$
46. $y = \frac{1}{1 + 2e^{-x}}$
47. $y = 2^{\sin x}$
48. $y = (\sqrt{3})^{\cos x}$
49. $y = 6^{\sec x}$
50. $y = 3^{\tan x}$
51. $y = 5xe^{3x}$
52. $y = x^3 e^{2x}$
53. $y = x^2 \sin(4x)$
54. $y = x^2 \cos(4x)$

In Problems 55–58, find y'. Treat a and b as constants.

55. $y = e^{-ax}\sin(bx)$
56. $y = e^{ax}\cos(-bx)$
57. $y = \frac{e^{ax} - 1}{e^{ax} + 1}$
58. $y = \frac{e^{-ax} + 1}{e^{bx} - 1}$

In Problems 59–62, write y as a function of x. Find $\frac{dy}{dx}$ using the Chain Rule.

59. $y = u^3$, $u = 3v^2 + 1$, $v = \frac{4}{x^2}$
60. $y = 3u$, $u = 3v^2 - 4$, $v = \frac{1}{x}$
61. $y = u^2 + 1$, $u = \frac{4}{v}$, $v = x^2$
62. $y = u^3 - 1$, $u = -\frac{2}{v}$, $v = x^3$

In Problems 63–70, find y'.

63. $y = e^{-2x}\cos(3x)$
64. $y = e^{\pi x}\tan(\pi x)$
65. $y = \cos(e^{x^2})$
66. $y = \tan(e^{x^2})$
67. $y = e^{\cos(4x)}$
68. $y = e^{\csc^2 x}$
69. $y = 4\sin^2(3x)$
70. $y = 2\cos^2(x^2)$

In Problems 71 and 72, find the derivative of each function by:
(a) Using the Chain Rule.
(b) Using the Power Rule for Functions.
(c) Expanding and then differentiating.
(d) Verify the answers from parts (a)–(c) are equal.

71. $y = (x^3+1)^2$
72. $y = (x^2-2)^3$

In Problems 73–78:
(a) Find an equation of the tangent line to the graph of f at the given point.
(b) Find an equation of the normal line to the graph of f at the given point.
(c) Use technology to graph f, the tangent line, and the normal line on the same screen.

73. $f(x)=(x^2-2x+1)^5$ at $(1, 0)$

74. $f(x)=(x^3-x^2+x-1)^{10}$ at $(0, 1)$

75. $f(x)=\dfrac{x}{(x^2-1)^3}$ at $\left(2, \dfrac{2}{27}\right)$

76. $f(x)=\dfrac{x^2}{(x^2-1)^2}$ at $\left(2, \dfrac{4}{9}\right)$

PAGE 226 77. $f(x)=\sin(2x)+\cos\dfrac{x}{2}$ at $(0, 1)$

78. $f(x)=\sin^2 x+\cos^3 x$ at $\left(\dfrac{\pi}{2}, 1\right)$

In Problems 79 and 80, find the indicated derivative.

79. $\dfrac{d^2}{dx^2}\cos(x^5)$ 80. $\dfrac{d^3}{dx^3}\sin^3 x$

PAGE 227 81. Suppose $h=f\circ g$. Find $h'(1)$ if $f'(2)=6$, $f(1)=4$, $g(1)=2$, and $g'(1)=-2$.

82. Suppose $h=f\circ g$. Find $h'(1)$ if $f'(3)=4$, $f(1)=1$, $g(1)=3$, and $g'(1)=3$.

83. Suppose $h=g\circ f$. Find $h'(0)$ if $f(0)=3$, $f'(0)=-1$, $g(3)=8$, and $g'(3)=0$.

84. Suppose $h=g\circ f$. Find $h'(2)$ if $f(1)=2$, $f'(1)=4$, $f(2)=-3$, $f'(2)=4$, $g(-3)=1$, and $g'(-3)=3$.

85. If $y=u^5+u$ and $u=4x^3+x-4$, find $\dfrac{dy}{dx}$ at $x=1$.

86. If $y=e^u+3u$ and $u=\cos x$, find $\dfrac{dy}{dx}$ at $x=0$.

Applications and Extensions

In Problems 87–94, find the indicated derivative.

87. $\dfrac{d}{dx}f(x^2+1)$ 88. $\dfrac{d}{dx}f(1-x^2)$

Hint: Let $u=x^2+1$.

89. $\dfrac{d}{dx}f\left(\dfrac{x+1}{x-1}\right)$ 90. $\dfrac{d}{dx}f\left(\dfrac{1-x}{1+x}\right)$

91. $\dfrac{d}{dx}f(\sin x)$ 92. $\dfrac{d}{dx}f(\tan x)$

93. $\dfrac{d^2}{dx^2}f(\cos x)$ 94. $\dfrac{d^2}{dx^2}f(e^x)$

95. **Rectilinear Motion** An object is in rectilinear motion and its position s, in meters, from the origin at time t seconds is given by $s=s(t)=A\cos(\omega t+\phi)$, where A, ω, and ϕ are constants.

(a) Find the velocity v of the object at time t.
(b) When is the velocity of the object 0?
(c) Find the acceleration a of the object at time t.
(d) When is the acceleration of the object 0?

96. **Rectilinear Motion** A bullet is fired horizontally into a bale of paper. The distance s (in meters) the bullet travels into the bale of paper in t seconds is given by

$$s=s(t)=8-(2-t)^3 \quad 0\le t\le 2$$

(a) Find the velocity v of the bullet at any time t.
(b) Find the velocity of the bullet at $t=1$.
(c) Find the acceleration a of the bullet at any time t.
(d) Find the acceleration of the bullet at $t=1$.
(e) How far into the bale of paper did the bullet travel?

97. **Rectilinear Motion** Find the acceleration a of a car if the distance s, in feet, it has traveled along a highway at time $t\ge 0$ seconds is given by

$$s(t)=\frac{80}{3}\left[t+\frac{3}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]$$

98. **Rectilinear Motion** An object moves in rectilinear motion so that at time $t\ge 0$ seconds, its position from the origin is $s(t)=\sin e^t$, in feet.

(a) Find the velocity v and acceleration a of the object at any time t.
(b) At what time does the object first have zero velocity?
(c) What is the acceleration of the object at the time t found in (b)?

PAGE 226 99. **Resistance** The resistance R (measured in ohms) of an 80-meter-long electric wire of radius x (in centimeters) is given by the formula $R=R(x)=\dfrac{0.0048}{x^2}$. The radius x is given by $x=0.1991+0.000003T$ where T is the temperature in Kelvin. How fast is R changing with respect to T when $T=320$ K?

100. **Pendulum Motion in a Car** The motion of a pendulum swinging in the direction of motion of a car moving at a low, constant speed can be modeled by

$$s=s(t)=0.05\sin(2t)+3t \qquad 0\le t\le\pi$$

where s is the distance in meters and t is the time in seconds.

(a) Find the velocity v at $t=\dfrac{\pi}{8}$, $t=\dfrac{\pi}{4}$, and $t=\dfrac{\pi}{2}$.
(b) Find the acceleration a at the times given in (a).
(c) Graph $s=s(t)$, $v=v(t)$, and $a=a(t)$ on the same screen.

Source: Mathematics students at Trine University, Angola, Indiana.

101. **Economics** The function $A(t)=102-90e^{-0.21t}$ represents the relationship between A, the percentage of the market penetrated by the latest generation smart phones, and t, the time in years, where $t=0$ corresponds to the year 2020.

(a) Find $\lim\limits_{t\to\infty}A(t)$ and interpret the result.
(b) Graph the function $A=A(t)$, and explain how the graph supports the answer in (a).

Ch. 3.2 Notes Implicit Differentiation

Explicit equations: (solved for y) Equations where x's and y's are separated on different sides of the equation: (example: $y = 9x^2 + 4\sqrt{x} + 3$)

Implicit equations: (not solved for y) Equations where x's and y's are mixed together on same side(s) of the equation (example: $y^2 = xy - x^2$)

Explicit Differentiation	Implicit Differentiation
$y = 3x^2 - 9x^3 + 5$	$y^2 - 5x = 4$

Steps:

1) Take derivative of each term with respect to x
2) If variable is y, find derivative and attach $\frac{dy}{dx}$ to the derivative
3) Move all terms containing $\frac{dy}{dx}$ to left side of equation.
4) Move all other terms to right side of equation.
5) Factor out $\frac{dy}{dx}$ on left side of equation
6) Solve for $\frac{dy}{dx}$

$\boxed{\text{Ex. 1}}$ $x^2 - 2y^3 + 4y = 2$ Find $\frac{dy}{dx}$

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$\boxed{\text{Ex. 2}}$ $3xy^3 - 2y = 7$ Find $\frac{dy}{dx}$ or y'

$\boxed{\text{Ex. 3}}$ Differentiate $y^2 = 5x$ with respect to t

Ch.3.2 Implicit Differentiation Worksheet #1

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Finding a Derivative In Exercises 1–16, find dy/dx by implicit differentiation.

Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f'g + fg'$$

1. $x^2 + y^2 = 9$

2. $x^2 - y^2 = 25$

4. $2x^3 + 3y^3 = 64$

5. $x^3 - xy + y^2 = 7$

6. $x^2y + y^2x = -2$

7. $x^3y^3 - y = x$

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Finding and Evaluating a Derivative In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

21. $xy = 6, \quad (-6, -1)$

22. $y^3 - x^2 = 4, \quad (2, 2)$

24. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$

25)

$(x^2 + 4)y = 8$

Point: $(2, 1)$

Find Equation of tangent line:

51. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

Ch. 3.2 Implicit Differentiation Worksheet #1

Key 13

Finding a Derivative In Exercises 1–16, find dy/dx by implicit differentiation.

Product Rule:
$$\frac{d}{dx}[f(x)g(x)] = f'g + fg'$$

1. $x^2 + y^2 = 9$

$2x + 2y\left(\frac{dy}{dx}\right) = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

2. $x^2 - y^2 = 25$

$2x - 2y\left(\frac{dy}{dx}\right) = 0$

$-2y\left(\frac{dy}{dx}\right) = -2x$

$\frac{dy}{dx} = \frac{-2x}{-2y}$

$\boxed{\frac{dy}{dx} = \frac{x}{y}}$

4. $2x^3 + 3y^3 = 64$

$6x^2 + 9y^2\left(\frac{dy}{dx}\right) = 0$

$9y^2\left(\frac{dy}{dx}\right) = -6x^2$

$\frac{dy}{dx} = \frac{-6x^2}{9y^2}$

$\boxed{\frac{dy}{dx} = \frac{-2x^2}{3y^2}}$

5. $x^3 - xy + y^2 = 7$ *product rule

$x^3 - \overset{f}{x}\overset{g}{y} + y^2 = 7$

$3x^2 - (\overset{f'}{(1)}\overset{g}{(y)} + \overset{f}{(x)}\overset{g'}{\left(\frac{dy}{dx}\right)}) + 2y\left(\frac{dy}{dx}\right) = 0$

$3x^2 - y - x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$

$-x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = y - 3x^2$

$\frac{dy}{dx}(-x + 2y) = y - 3x^2$

$\boxed{\frac{dy}{dx} = \frac{y - 3x^2}{-x + 2y}}$

6. $x^2y + y^2x = -2$

$\overset{f}{x^2}\overset{g}{y} + \overset{f}{y^2}\cdot\overset{g}{x} = -2$

$\overset{f'}{2x}\cdot\overset{g}{y} + \overset{f}{x^2}\cdot\overset{g'}{\frac{dy}{dx}} + \overset{f'}{2y\left(\frac{dy}{dx}\right)}\cdot\overset{g}{x} + \overset{f}{y^2}\cdot\overset{g'}{(1)} = 0$

$2xy + x^2\left(\frac{dy}{dx}\right) + 2xy\left(\frac{dy}{dx}\right) + y^2 = 0$

$x^2\left(\frac{dy}{dx}\right) + 2xy\left(\frac{dy}{dx}\right) = -2xy - y^2$

$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$

$\boxed{\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}}$

7. $x^3y^3 - y = x$

$\overset{f}{x^3}\overset{g}{y^3} - y = x$

$\overset{f'}{3x^2}\cdot\overset{g}{y^3} + \overset{f}{x^3}\cdot\overset{g'}{3y^2\left(\frac{dy}{dx}\right)} - 1\left(\frac{dy}{dx}\right) = 1$

$3x^2y^3 + 3x^3y^2\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 1$

$3x^3y^2\left(\frac{dy}{dx}\right) - 1\left(\frac{dy}{dx}\right) = 1 - 3x^2y^3$

$\frac{dy}{dx}(3x^3y^2 - 1) = 1 - 3x^2y^3$

$\boxed{\frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}}$

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Finding and Evaluating a Derivative In Exercises 21–28, find dy/dx by implicit differentiation and evaluate the derivative at the given point.

21. $xy = 6, \quad (-6, -1)$

$\overbrace{x}^{f}\overbrace{y}^{g} = 6$

$\overbrace{(1)}^{f'}\overbrace{(y)}^{g} + \overbrace{(x)}^{f}\overbrace{\left(\frac{dy}{dx}\right)}^{g'} = 0$

$x\left(\frac{dy}{dx}\right) = -y$

$\frac{dy}{dx} = \frac{-y}{x}$

$\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-(-1)}{(-6)}$

$\boxed{\left.\frac{dy}{dx}\right|_{(-6,-1)} = \frac{-1}{6}}$

22. $y^3 - x^2 = 4, \quad (2, 2)$

$3y^2\left(\frac{dy}{dx}\right) - 2x = 0$

$3y^2\left(\frac{dy}{dx}\right) = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2}$

$\left.\frac{dy}{dx}\right|_{(2,2)} = \frac{2(2)}{3(2)^2} = \boxed{\frac{1}{3}}$

24. $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\left(\frac{dy}{dx}\right) = 0$

$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}}\left(\frac{dy}{dx}\right) = 0$

$\frac{2}{3y^{1/3}}\left(\frac{dy}{dx}\right) = \frac{-2}{3x^{1/3}}$

$\frac{dy}{dx} = \frac{-2}{3x^{1/3}} \cdot \frac{3y^{1/3}}{2}$

$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$

$\left.\frac{dy}{dx}\right|_{(8,1)} = \frac{-(1)^{1/3}}{(8)^{1/3}}$

$\left.\frac{dy}{dx}\right|_{(8,1)} = \boxed{\frac{-1}{2}}$

25)

$(x^2 + 4)y = 8$

Point: (2, 1)

$\overbrace{(x^2+4)}^{f}\overbrace{(y)}^{g} = 8$

$\overbrace{2x}^{f'} \cdot \overbrace{y}^{g} + \overbrace{(x^2+4)}^{f}\overbrace{\left(\frac{dy}{dx}\right)}^{g'} = 0$

$2xy + (x^2+4)\left(\frac{dy}{dx}\right) = 0$

$(x^2+4)\left(\frac{dy}{dx}\right) = -2xy$

$\frac{dy}{dx} = \frac{-2xy}{x^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2xy}{x^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2(2)(1)}{2^2+4}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-4}{8}$

$= \boxed{\frac{-1}{2}}$

Find Equation of tangent line:

51. $\sqrt{x} + \sqrt{y} = 5, \quad (9, 4)$

$x^{1/2} + y^{1/2} = 5$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\left(\frac{dy}{dx}\right) = 0$

$\frac{1}{2x^{1/2}} + \frac{1}{2y^{1/2}}\left(\frac{dy}{dx}\right) = 0$

$\frac{1}{2y^{1/2}}\left(\frac{dy}{dx}\right) = \frac{-1}{2x^{1/2}}$

$\frac{dy}{dx} = \frac{-1}{2x^{1/2}} \cdot \frac{2y^{1/2}}{1}$

$\frac{dy}{dx} = \frac{-2y^{1/2}}{2x^{1/2}}$

$\frac{dy}{dx} = \frac{-y^{1/2}}{x^{1/2}}$

$\left.\frac{dy}{dx}\right|_{(9,4)} = \frac{-(4)^{1/2}}{(9)^{1/2}} = \frac{-2}{3}$

point: (9, 4)

slope: $m = \frac{-2}{3}$

$y - y_1 = m(x - x_1)$

$\boxed{y - 4 = \frac{-2}{3}(x - 9)}$

15

AP Calculus AB–5 / BC–5

2000

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

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3.2 Implicit Differentiation Practice FRQ

AP Calculus AB–5 / BC–5 **2000**

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

* Implicit differentiation
* product Rule

a) $\overbrace{x}^{f}\overbrace{y^2}^{g} - (\overbrace{x^3}^{f}\overbrace{y}^{g}) = 6$

$\overbrace{(1)}^{f'}\cdot\overbrace{(y^2)}^{g} + \overbrace{(x)}^{f}\overbrace{2y\left(\frac{dy}{dx}\right)}^{g'} - (\overbrace{3x^2}^{f'}\cdot\overbrace{y}^{g} + \overbrace{x^3}^{f}\cdot\overbrace{\left(\frac{dy}{dx}\right)}^{g'}) = 0$

$y^2 + \underline{2xy\left(\frac{dy}{dx}\right)} - 3x^2y - \underline{x^3\left(\frac{dy}{dx}\right)} = 0$

$\frac{dy}{dx}(2xy - x^3) = 3x^2y - y^2$

$\boxed{\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}}$

b) * $x = 1 \rightarrow xy^2 - x^3y = 6 \rightarrow (1)y^2 - (1)^3y = 6 \rightarrow y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3, y = -2$

points: $(1,3)$ and $(1,-2)$

$\left.\frac{dy}{dx}\right|_{(1,3)} = \frac{3(3) - 3^2}{2(1)(3) - 1} = 0$

$y - 3 = 0(x-1)$

$\boxed{y = 3}$

$\left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{3(-2) - (-2)^2}{2(1)(-2) - 1} = \frac{-10}{-5} = 2$

$\boxed{y + 2 = 2(x-1)}$

c) vertical tangent set denominator of $\frac{dy}{dx} = 0$

$2xy - x^3 = 0$

$x(2y - x^2) = 0$

$\cancel{x = 0}$ extraneous solution

$2y - x^2 = 0$

$2y - x^2 = 0$

$y = \frac{x^2}{2}$

$xy^2 - x^3y = 6$

$x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6$

$\frac{x^5}{4} - \frac{x^5}{2} = 6$

$\frac{x^5}{4} - \frac{2x^5}{4} = 6 \rightarrow -\frac{x^5}{4} = 6$

$-x^5 = 24$

$x^5 = -24$

$\boxed{x = \sqrt[5]{-24}}$

(17)

EXAMPLE 7 **Differentiating Functions Using the Power Rule**

PAGE 238

(a) $\frac{d}{ds}(s^3-2s+1)^{5/3} = \frac{5}{3}(s^3-2s+1)^{2/3}\frac{d}{ds}(s^3-2s+1) = \frac{5}{3}(s^3-2s+1)^{2/3}(3$

(b) $\frac{d}{dx}\sqrt[3]{x^4-3x+5} = \frac{d}{dx}(x^4-3x+5)^{1/3} = \frac{1}{3}(x^4-3x+5)^{-2/3}\frac{d}{dx}(x^4-3x+5$

$$= \frac{4x^3-3}{3(x^4-3x+5)^{2/3}}$$

(c) $\frac{d}{d\theta}[\tan(3\theta)]^{-3/4} = -\frac{3}{4}[\tan(3\theta)]^{-7/4}\frac{d}{d\theta}\tan(3\theta) = -\frac{3}{4}[\tan(3\theta)]^{-7/4}\cdot\sec^2(3\theta)\cdot$

$$= -\frac{9\sec^2(3\theta)}{4[\tan(3\theta)]^{7/4}}$$

NOW WORK Problem 39 and AP® Practice Problems 2, 4, and 7.

3.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* Implicit differentiation is a technique for finding the derivative of an implicitly defined function.
2. *True or False* If $y^q = x^p$ for integers p and q, then $qy^{q-1} = px^{p-1}$.
3. $\frac{d}{dx}(3x^{1/3}) =$ ________.
4. If $y = (x^2+1)^{3/2}$, then $y' =$ ________.

Skill Building

In Problems 5–30, find $y' = \frac{dy}{dx}$ using implicit differentiation.

5. $3x+2y=3$
6. $2x-5y=7$
7. $x^2+y^2=4$
8. $y^4-4x^2=4$
9. $e^y=\sin x$
10. $e^y=\tan x$
11. $e^{x+y}=y$
12. $e^{x+y}=x^2$
13. $x^2y=5$
14. $x^3y=8$
15. $x^2-y^2-xy=2$ (PAGE 238)
16. $x^2-4xy+y^2=y$
17. $\frac{1}{x}+\frac{1}{y}=1$ (PAGE 237)
18. $\frac{1}{x}-\frac{1}{y}=4$
19. $x^2+y^2=\frac{2y}{x}$
20. $x^2+y^2=\frac{2y^2}{x^2}$
21. $e^x\sin y+e^y\cos x=4$
22. $e^y\cos x+e^{-x}\sin y=10$
23. $(x^2+y)^3=y$
24. $(x+y^2)^3=3x$
25. $y=\tan(x-y)$ (PAGE 239)
26. $y=\cos(x+y)$
27. $y=x\sin y$
28. $y=x\cos y$
29. $x^2y=e^{xy}$
30. $ye^x=y-x$

In Problems 31–46, find y'.

31. $y=x^{2/3}+4$ (PAGE 241)
32. $y=x^{1/3}-1$
33. $y=\sqrt[3]{x^2}$
34. $y=\sqrt[4]{x^5}$
35. $y=\sqrt[3]{x}-\frac{1}{\sqrt[3]{x}}$
36. $y=\sqrt{x}+\frac{1}{\sqrt{x}}$
37. $y=(x^3-1)^{1/2}$
38. $y=(x^2-1)^{1/3}$
39. $y=x\sqrt{x^2-1}$ (PAGE 242)
40. $y=x\sqrt{x^3+1}$
41. $y=e^{\sqrt{x^2-9}}$
42. $y=\sqrt{e^x}$
43. $y=(x^2\cos x)^{3/2}$
44. $y=(x^2\sin x)^{3/2}$
45. $y=(x^2-3)^{3/2}(6x+1)^{5/3}$
46. $y=\frac{(2x^3-1)^{4/3}}{(3x+4)^{5/2}}$

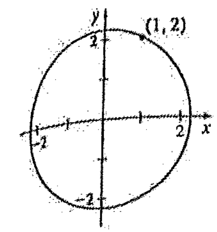
In Problems 47–52, find y' and y''.

47. $x^2+y^2=4$
48. $x^2-y^2=1$
49. $x^2-y^2=4+5x$ (PAGE 240)
50. $4xy=x^2+y^2$
51. $y=\sqrt{x^2+1}$
52. $y=\sqrt{4-x^2}$

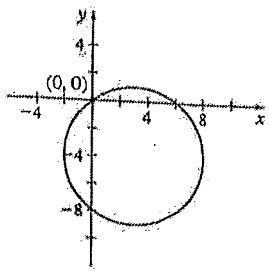
In Problems 53–58, for each implicitly defined equation:
(a) Find the slope of the tangent line to the graph of the equation at the indicated point.
(b) Write an equation for this tangent line.
(c) Graph the tangent line on the same axes as the graph of the equation.

(18)

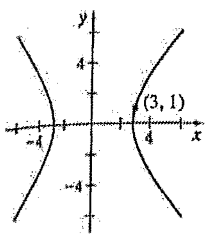
53. $x^2+y^2=5$ at $(1, 2)$



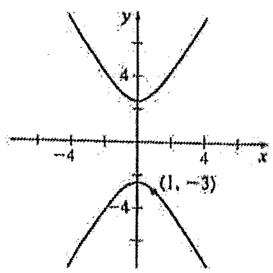
54. $(x-3)^2+(y+4)^2=25$ at $(0, 0)$



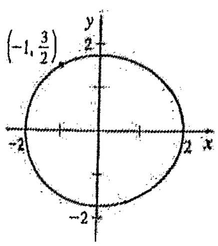
55. $x^2-y^2=8$ at $(3, 1)$



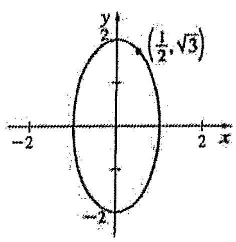
56. $y^2-3x^2=6$ at $(1, -3)$



57. $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$ at $\left(-1, \dfrac{3}{2}\right)$



58. $x^2+\dfrac{y^2}{4}=1$ at $\left(\dfrac{1}{2}, \sqrt{3}\right)$



59. Find y' and y'' at the point $(-1, 1)$ on the graph of

$$3x^2y+2y^3=5x^2$$

60. Find y' and y'' at the point $(0, 0)$ on the graph of

$$4x^3+2y^3=x+y$$

Applications and Extensions

In Problems 61–68, find y'.

Hint: Use the fact that $|x|=\sqrt{x^2}$.

61. $y=|3x|$

62. $y=|x^5|$

63. $y=|2x-1|$

64. $y=|5-x^2|$

65. $y=|\cos x|$

66. $y=|\sin x|$

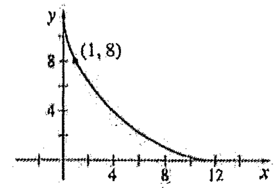
67. $y=\sin|x|$

68. $|x|+|y|=1$

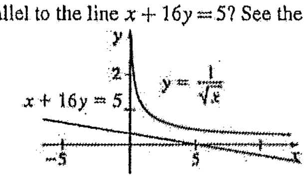
69. **Tangent Line to a Hypocycloid** The graph of

$$x^{2/3}+y^{2/3}=5$$

is called a **hypocycloid**. Part of its graph is shown in the figure. Find an equation of the tangent line to the hypocycloid at the point $(1, 8)$.



70. **Tangent Line** At what point does the graph of $y=\dfrac{1}{\sqrt{x}}$ have a tangent line parallel to the line $x+16y=5$? See the figure.

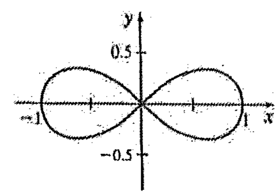


71. **Tangent Line** For the equation $x+xy+2y^2=6$:
   (a) Find an expression for the slope of the tangent line at any point (x, y) on the graph.
   (b) Write an equation for the line tangent to the graph at the point $(2, 1)$.
   (c) Find the coordinates of any other point on this graph with slope equal to the slope at $(2, 1)$.
   CAS (d) Graph the equation and the tangent lines found in parts (b) and (c) on the same screen.

72. **Tangent Line to a Lemniscate** The graph of

$$(x^2+y^2)^2=x^2-y^2$$

called a **lemniscate**, is shown in the figure. There are exactly four points at which the tangent line to the lemniscate is horizontal. Find them.



73. **Rectilinear Motion** An object of mass m moves in rectilinear motion so that at time $t>0$ its position s from the origin and its velocity $v=\dfrac{ds}{dt}$ satisfy the equation

$$m\left(v^2-v_0^2\right)=k\left(s_0^2-s^2\right)$$

where k is a positive constant and v_0 and s_0 are the initial velocity and position, respectively, of the object. Show that if $v>0$, then

$$ma=-ks$$

where $a=\dfrac{d^2s}{dt^2}$ is the acceleration of the object.

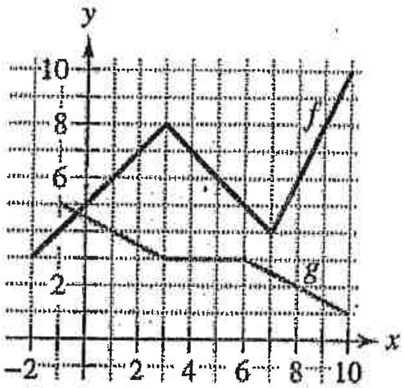
Hint: Differentiate the expression $m\left(v^2-v_0^2\right)=k\left(s_0^2-s^2\right)$ with respect to t.

Ch.3.1 **Product,Quotient Rule** HW Problems Evaluating Derivatives using graphs

Evaluating Derivatives **In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$.**

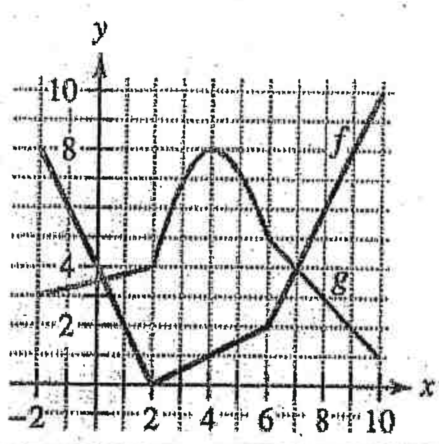
81. (a) Find $p'(1)$.

(b) Find $q'(4)$.



82. (a) Find $p'(4)$.

(b) Find $q'(7)$.



Using Relationships **In Exercises 103–106, use the given information to find $f'(2)$.**

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

103. $f(x) = 2g(x) + h(x)$

105. $f(x) = \dfrac{g(x)}{h(x)}$

20

Ch.2.4 p.139

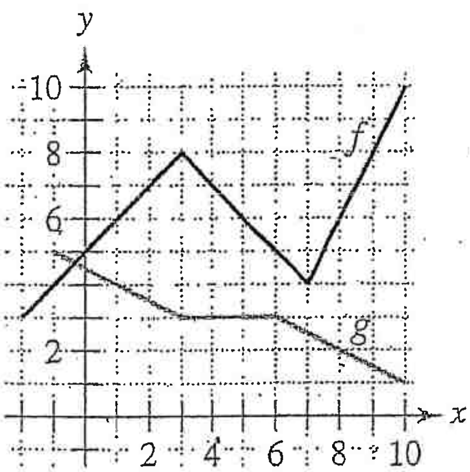
[redacted]

20

In Exercises 99 and 100, the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

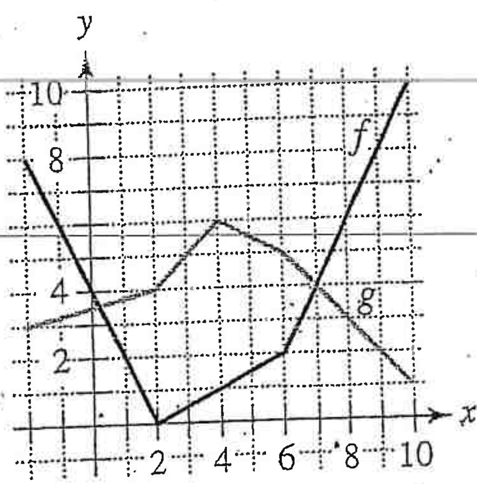
99. (a) Find $h'(1)$.

(b) Find $s'(5)$.



100. (a) Find $h'(3)$.

(b) Find $s'(9)$.



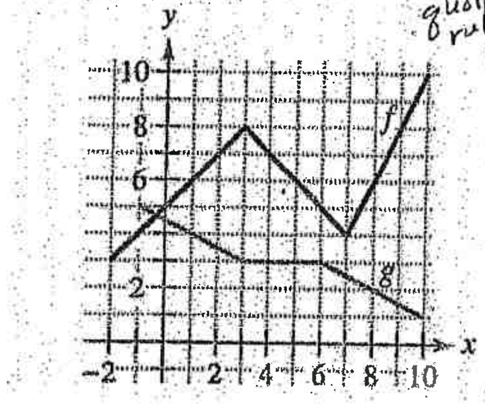
Product,Quotient Rule HW Problems   Evaluating Derivatives using graphs   Key (21)

Evaluating Derivatives In Exercises 81 and 82, use the graphs of f and g. Let $p(x) = f(x)g(x)$ and $q(x) = f(x)/g(x)$. ↙ *product rule

81. (a) Find $p'(1)$.

(b) Find $q'(4)$.

quotient rule



a) $p'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$p'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1)$

$= (1)\cdot(4) + (6)\cdot(-\tfrac{1}{2}) = 4-3 = \boxed{1}$

b) $q'(x) = \dfrac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$

$q'(4) = \dfrac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2}$

$q'(4) = \dfrac{(-1)(3)-(7)(0)}{3^2}$

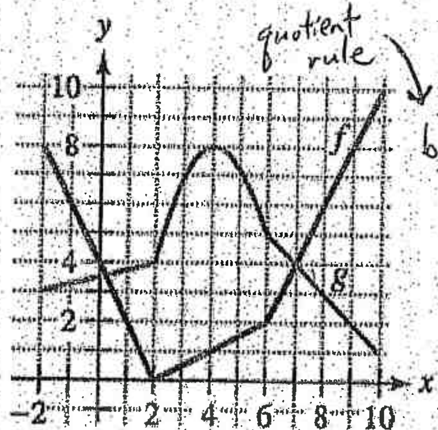
$q'(4) = \dfrac{-3}{3^2} = \boxed{-\dfrac{1}{3}}$

---

82. (a) Find $p'(4)$.

(b) Find $q'(7)$.

quotient rule



product Rule → a) $p'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$p'(4) = f'(4)\cdot g(4) + f(4)\cdot g'(4)$

$= (\tfrac{1}{2})\cdot(8) + (1)(0) = 4$

$\boxed{p'(4) = 4}$

b) $q'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$q'(7) = \dfrac{f'(7)g(7) - f(7)g'(7)}{g(7)^2}$

$q'(7) = \dfrac{(2)(4) - 4(-1)}{4^2} = \dfrac{8+4}{16} = \dfrac{12}{16} = \dfrac{3}{4}$

$\boxed{q'(7) = 3/4}$

Using Relationships In Exercises 103–106, use the given information to find $f'(2)$.

$g(2) = 3$ and $g'(2) = -2$

$h(2) = -1$ and $h'(2) = 4$

103. $f(x) = 2g(x) + h(x)$

$f'(x) = 2g'(x) + h'(x)$

$f'(2) = 2g'(2) + h'(2)$

$= 2(-2) + 4 = 0$

$\boxed{f'(2) = 0}$

Apply quotient rule

105. $f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$

$f'(2) = \dfrac{g'(2)h(2) - g(2)h'(2)}{h(2)^2}$

$f'(2) = \dfrac{(-2)(-1) - 3(4)}{(-1)^2}$

$f'(2) = \dfrac{2-12}{1} = -10$

$\boxed{f'(2) = -10}$

p139

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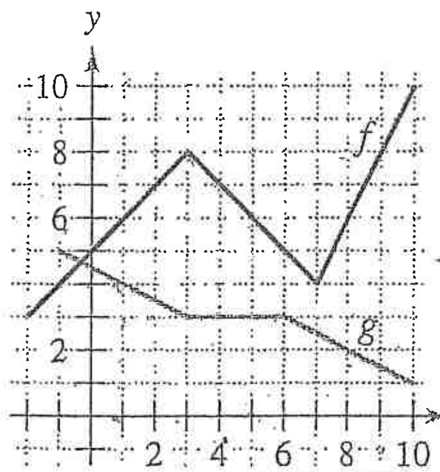
In Exercises 99 the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.

$h(x) = f(g(x))$
$s(x) = g(f(x))$

99. (a) Find $h'(1)$.
(b) Find $s'(5)$.



(a) $h'(x) = f'[g(x)] \cdot g'(x)$
$h'(1) = f'[g(1)] \cdot g'(1)$
$h'(1) = f'[4] \cdot (-\frac{1}{2})$
$= (-1)(-\frac{1}{2})$
$\boxed{h'(1) = \frac{1}{2}}$

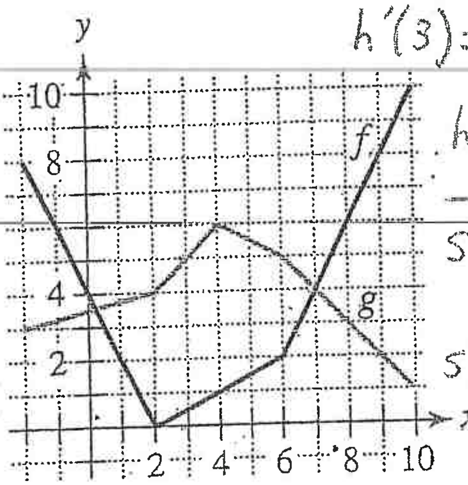
$g(1) = 4$
$g'(1) = -\frac{1}{2}$
$f'(4) = -1$

$s'(x) = g'(f(x)) \cdot f'(x)$
$s'(5) = g'(f(5)) \cdot f'(5)$
$s'(5) = g'(6) \cdot (-1)$
$\boxed{s'(5) = \text{DNE}}$

$f(5) = 6$
$f'(5) = -1$
$g'(6) = \text{DNE}$

$h(x) = f(x)g(x)$
$s(x) = \frac{f(x)}{g(x)}$

100. (a) Find $h'(3)$.
(b) Find $s'(9)$.



$h'(x) = f'(x)g(x) + f(x)g'(x)$
$h'(3) = f'(3)g(3) + f(3)g'(3)$
$h'(3) = (\frac{1}{2})(5) + (\frac{1}{2})(1)$
$h'(3) = \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = \boxed{3}$

$f(3) = \frac{1}{2}$
$f'(3) = \frac{1}{2}$
$g(3) = 5$
$g'(3) = 1$

$s'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$s'(9) = \frac{f'(9)g(9) - f(9)g'(9)}{[g(9)]^2}$

$f(9) = 8$
$f'(9) = 2$
$g(9) = 2$
$g'(9) = -1$

$s'(9) = \frac{(2)(2) - (8)(-1)}{(2)^2} = \frac{4+8}{4} = \frac{12}{4} = 3$ $\boxed{s'(9) = 3}$

Ch. 3.1 Chain Rule HW Problems #102, #115

102. Using Relationships Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

Recall: Product Rule: $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx} f[g(x)] = f'[g(x)] * g'(x)$

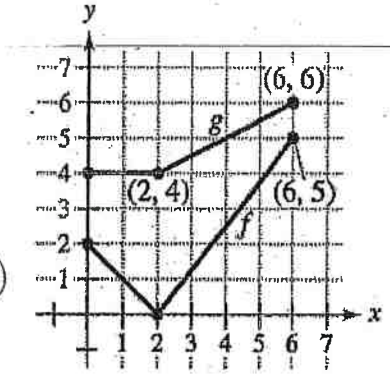
(a) $f(x) = g(x)h(x)$

(b) $f(x) = g(h(x))$

(c) $f(x) = \dfrac{g(x)}{h(x)}$

(d) $f(x) = [g(x)]^3$

115. Think About It Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



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(4)

C Implicit Differentiation Vertical, Horizontal Tangent Lines HW Problems #57, #58

Vertical and Horizontal Tangent Lines **In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line.**

*Find **Horizontal Tangent** lines by setting **numerator of derivative** equal to zero, solve for x

*Find **Vertical Tangent** lines by setting **denominator of derivative** equal to zero, solve for x

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

(25) Key

Chain Rule HW Problems #102, #115

102. Using Relationships Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

Recall: product rule: $\frac{d}{dx} f(x)g(x) = f'(x)(g(x)) + f(x)g'(x)$

quotient rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Chain rule: $\frac{d}{dx} f[g(x)] = f'(g(x)) \cdot g'(x)$

(a) $f(x) = g(x)h(x)$ *product rule

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$f'(5) = g'(5)h(5) + g(5)h'(5)$

$f'(5) = 6(3) + (-3)(-2)$

$f'(5) = 18 + 6 = 24$

$\boxed{f'(5) = 24}$

(b) $f(x) = g(h(x))$ *chain rule

$f'(x) = g'[h(x)] \cdot h'(x)$

$f'(5) = g'[h(5)] \cdot h'(5)$

$= g'(3) \cdot h'(5)$

$= g'(3) \cdot -2$

$\boxed{f'(5) = -2g'(3)}$

(c) $f(x) = \dfrac{g(x)}{h(x)}$ *quotient rule

$f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$

$f'(5) = \dfrac{g'(5)h(5) - g(5)h'(5)}{h(5)^2}$

$= \dfrac{6(3) - (-3)(-2)}{3^2}$

$= \dfrac{18-6}{9} = \dfrac{12}{9} = 4/3$

$\boxed{f'(5) = 4/3}$

(d) $f(x) = [g(x)]^3$ *chain rule

$f'(x) = 3[g(x)]^2 \cdot g'(x)$

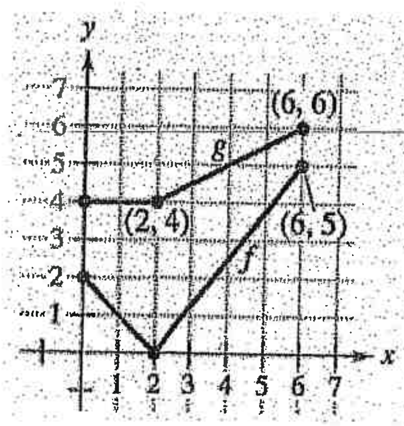
$f'(5) = 3[g(5)]^2 \cdot g'(5)$

$= 3[-3]^2 \cdot 6$

$= 3(9)(6) = 162$

$\boxed{f'(5) = 162}$

115. Think About It Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$. ← Apply chain rule



a) $r'(x) = f'[g(x)] \cdot g'(x)$

$r'(1) = f'[g(1)] \cdot g'(1)$

$r'(1) = f'[4] \cdot 0$

$r'(1) = \frac{5}{4}(0) = \boxed{0}$

$\boxed{r'(1) = 0}$

b) $s'(x) = g'[f(x)] \cdot f'(x)$

$s'(4) = g'[f(4)] \cdot f'(4)$

$= g'[5/2] \cdot (5/4)$

$= (\frac{1}{2})(5/4) = 5/$

$\boxed{s'(4) = 5/8}$

(26) (26) Key

Implicit Differentiation Vertical, Horizontal Tangent Lines HW Problems #57, #58

Vertical and Horizontal Tangent Lines In Exercises 57 and 58, find the points at which the graph of the equation has a vertical or horizontal tangent line. $\left(\frac{dy}{dx}\right)$

*Find **Horizontal Tangent** lines by setting **numerator of derivative** equal to zero, solve for x

*Find **Vertical Tangent** lines by setting **denominator of derivative** equal to zero, solve for x

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$ ← Apply implicit differentiation

$50x + 32y\left(\frac{dy}{dx}\right) + 200 - 160\left(\frac{dy}{dx}\right) + 0 = 0$

$32y\left(\frac{dy}{dx}\right) - 160\left(\frac{dy}{dx}\right) = -50x - 200$

$\frac{dy}{dx}\left[32y - 160\right] = -50x - 200$

$\frac{dy}{dx} = \frac{-50x - 200}{32y - 160}$

horiz. tangent: $-50x - 200 = 0$

$-50x = +200$

plug into equation $\boxed{x = -4}$

$25(-4)^2 + 16y^2 + 200(-4) - 160y + 400 = 0$

$400 + 16y^2 - 800 - 160y + 400 = 0$

$16y^2 - 160y = 0 \quad 16y(y-10) = 0 \quad y = 0, 10$

$\boxed{\text{Horiz. tangents: } (-4,0) \text{ and } (-4,10)}$

vertical tangent: $32y - 160 = 0 \quad 32y = 160 \quad \underline{y = 5}$

$25x^2 + 16(5)^2 + 200x - 160(5) + 400 = 0$

$25x^2 + 400 + 200x - 800 + 400 = 0$

$25x^2 + 200x = 0$

$25x(x+8) = 0$

$x = 0, -8$

$\boxed{\text{vertical tangents: } (0,5) \text{ and } (-8,5)}$

58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2y\left(\frac{dy}{dx}\right) - 8 + 4\left(\frac{dy}{dx}\right) + 0 = 0$

$2y\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right) = 8 - 8x$

$\frac{dy}{dx}(2y + 4) = 8 - 8x$

$\boxed{\frac{dy}{dx} = \frac{8 - 8x}{2y + 4}}$

horizontal tangent: $8 - 8x = 0 \quad \underline{x = 1}$

$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$y^2 + 4y = 0$

$y(y+4) = 0$

$y = 0, -4$

$\boxed{\text{horizontal tangents: } (1,0) \text{ and } (1,-4)}$

vertical tangents: $2y + 4 = 0$

$2y = -4 \quad \underline{y = -2}$

$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$4x^2 + 4 - 8x - 8 + 4 = 0$

$4x^2 - 8x = 0$

$4x(x-2) = 0$

$x = 0, 2$

$\boxed{\text{vertical tangents } (0,-2) \text{ and } (2,-2)}$

Review (Differentiable Piecewise Functions)

Finding Constants to make a Piecewise Function continuous and differentiable.

2. How to find constants a *and* b so that the function is continuous and differentiable.
   a. You will be given a function like this one:

$$f(x) = \begin{cases} ax + 3 & x \leq 1 \\ 3x^2 + x + b & x > 1 \end{cases}$$

3. Practice find constants a and b so that the function is continuous and differentiable.

   a. $f(x) = \begin{cases} 2x^2 + 2x + a & x \leq -1 \\ -2bx + 1 & x > -1 \end{cases}$

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Review (Differentiable Piecewise Functions)

Finding Constants to make a Piecewise Function continuous and differentiable.

2. **How to find constants a and b so that the function is continuous and differentiable.**
   a. **You will be given a function like this one:**

$$f(x) = \begin{cases} ax+3 & x \le 1 \\ 3x^2+x+b & x > 1 \end{cases}$$

continuous - share same y-value (set equations equal) at $x=1$

$ax+3 = 3x^2+x+b$

$a(1)+3 = 3(1)^2+1+b$

$a+3 = 4+b$

$a = 1+b$

$7 = 1+b$

$\boxed{6=b}$

differentiable - share same slope (set derivatives equal) at $x=1$

$$f'(x) = \begin{cases} a, & x \le 1 \\ 6x+1, & x<1 \end{cases}$$

$a = 6x+1$

$a = 6(1)+1$

$\boxed{a=7}$

3. **Practice find constants a and b so that the function is continuous and differentiable.**

a. $f(x) = \begin{cases} 2x^2+2x+a & x \le -1 \\ -2bx+1 & x > -1 \end{cases}$

continuous (at $x=-1$)

$2x^2+2x+a = -2bx+1$

$2(-1)^2+2(-1)+a = -2b(-1)+1$

$a = 2b+1$

$a = 2(1)+1$

$\boxed{a=3}$

differentiable at $x=-1$

$$f'(x) = \begin{cases} 4x+2 & x \le -1 \\ -2b & x > -1 \end{cases}$$

$4x+2 = -2b$

$4(-1)+2 = -2b$

$-2 = -2b$

$\boxed{1=b}$

(29)

AB Calculus Ch 3 Test Review

1. The average rate of change of $f(x) = 4x - x^2$ on the interval [1,3] is _______.

a.) -2 b.) -1 c.) 0 d.) 1 e.) 2

2. The instantaneous rate of change of f(x) at the endpoints are : ______________________

3. Find $\frac{d^2y}{dx^2}$ for $y = \sqrt{(x^2+1)}$.

a.) $\frac{1}{(x^2+1)^{\frac{3}{2}}}$ b.) $\frac{1}{(x^2+1)^{\frac{1}{2}}}$ c.) $\frac{x}{(x^2+1)^{\frac{1}{2}}}$ d.) $\frac{x}{(x^2+1)^{\frac{3}{2}}}$ e.) $\frac{1}{2(x^2+1)^{\frac{3}{2}}}$

4. If the line tangent to the graph of the function f at the point (1,7) passes through the point (-2,-2), then $f'(1)$ =

a.) - 5 b.) 1 c.) 3 d.) 7 e.) undefined

5. Find a and b such that

$$f(x) = \begin{cases} ax^3 & x \le 3 \\ x^2 + b & x > 3 \end{cases}$$ is differentiable everywhere.

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6. A particle moves along a straight line according to the given equation: $s(t)=\frac{2}{3}t^3-2t^2-1,\ t\geq 0$.

a) Find when the particle is moving to the left.

b.) Find when the velocity is increasing.

c.) Find when the speed is decreasing.

d.) Where is the particle located when the velocity is zero?

e) Find particle's displacement from t = 0 to t = 3

f) Find particle's distance from t = 0 to t = 3

g) Is the velocity increasing or decreasing at t=2?

h) Find the velocity and position when acceleration is zero.

7. Find $\lim_{h\to 0}\frac{2(-1+h)^3-3(-1+h)-1}{h}$

8. If $f(x)=x^4-4x$, evaluate $\lim_{x\to -1}\frac{f(x)-f(-1)}{x+1}$.

a.) - 8 b.) 0 c.) 1 d.) 2 e.) 4

9. Find the points at which the graph of the equation has a vertical or horizontal tangent line:

$4x^2+y^2-8x+4y+4=0$

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AB Calculus Ch3 Test Review Session

1. The average rate of change of $f(x)=4x-x^2$ on the interval [1,3] is________.

a.) -2 b.) -1 $\boxed{\text{c.) } 0}$ d.) 1 e.) 2

$f(1)=4-1=3$

$f(3)=4(3)-9=3$

$\frac{f(3)-f(1)}{3-1}=\frac{3-3}{2}=\frac{0}{2}$

2. The instantaneous rate of change of f(x) at the endpoints are :

$f'(x)=4-2x$

$f'(1)=4-2(1)=2$

$f'(3)=4-2(3)=-2$

$\boxed{\begin{array}{l} f'(1)=2 \\ f'(3)=-2 \end{array}}$

3. Find $\frac{d^2y}{dx^2}$ for $y=\sqrt{(x^2+1)}$. $=(x^2+1)^{1/2}$ $f'(x)=\frac{1}{2}(x^2+1)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+1}}$

$\boxed{\text{a.)}}$ $\frac{1}{(x^2+1)^{\frac{3}{2}}}$ b.) $\frac{1}{(x^2+1)^{\frac{1}{2}}}$ c.) $\frac{x}{(x^2+1)^{\frac{1}{2}}}$ d.) $\frac{x}{(x^2+1)^{\frac{3}{2}}}$ e.) $\frac{1}{2(x^2+1)^{\frac{3}{2}}}$

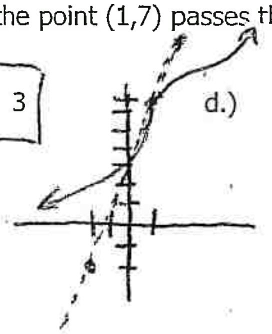
$f'(x)=\frac{x}{\sqrt{x^2+1}}=\frac{x}{(x^2+1)^{1/2}}$

$f''(x)=\frac{1(x^2+1)^{1/2}-x\cdot\frac{1}{2}(x^2+1)^{-1/2}(2x)}{x^2+1}$

$=\frac{\sqrt{x^2+1}-\frac{x^2}{\sqrt{x^2+1}}}{x^2+1}\cdot\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$ $\rightarrow \frac{x^2+1-x^2}{(x^2+1)^{3/2}}=\boxed{\frac{1}{(x^2+1)^{3/2}}}$

4. If the line tangent to the graph of the function f at the point (1,7) passes through the point (-2,-2), then $f'(1)=$

a.) -5 b.) 1 $\boxed{\text{c.) } 3}$ d.) 7 e.) undefined



Find slope : $m=\frac{-2-7}{-2-1}=\frac{-9}{-3}=3$

5. Find a and b such that

$$f(x)=\begin{cases} ax^3 & x\le 3 \\ x^2+b & x>3 \end{cases}$$ is differentiable everywhere.

↳ continuous and differentiable

$\boxed{\text{at } x=3}$

$ax^3=x^2+b$

$a(3)^3=3^2+b$

$27a=9+b$

$27(2/9)=9+b$

$6=9+b$

$\boxed{-3=b}$

at $x=3$

$3ax^2=2x$

$3a(9)=2(3)$

$\boxed{a=2/9}$

32

$v(t) = 2t^2 - 4t$ $v(t)$: $-$ on $(0, 2)$, $+$ after 2 $0 = 2t(t-2)$

$a(t) = 4t - 4$ $t = 0, 2$

6. A particle moves along a straight line according to the given equation: $s(t) = \frac{2}{3}t^3 - 2t^2 - 1$, $t \ge 0$.

a) Find when the particle is moving to the left.

$\boxed{(0,2)}$ $0 < t < 2$

b.) Find when the velocity is increasing. $\boxed{a(t) > 0}$

$a(t)$: $-$ on $(0, 1)$, $+$ after 1 $\boxed{(1, \infty)}$

$0 = 4t - 4$ $0 = 4(t-1)$

c.) Find when the speed is decreasing.

opposite signs for $v(t)$ and $a(t)$

$\boxed{(1,2)}$ $1 < t < 2$

d.) Where is the particle located when the velocity is zero?

$t = 0, 2$ $s(2) = \frac{2}{3}(2)^3 - 2(2)^2 - 1$

$= \frac{16}{3} - 8 - 1 = \frac{16}{3} - \frac{27}{3} = \frac{-11}{3}$

$\boxed{s(0) = -1 \quad s(2) = -11/3}$

e) Find particle's displacement from t = 0 to t = 3

$s(0) = -1$ displacement $= 0$

$s(3) = -1$

f) Find particle's distance from t = 0 to t = 3

$s(0) = -1$ $\}\ 8/3$

$s(2) = -11/3$ $\}\ 8/3$ $= \frac{16}{3} \approx 5.3\overline{3}$

$s(3) = -1$

g) Is the velocity increasing or decreasing at t=2?

$a(2) = 8 - 4 = 4$ since $a(2) > 0$ velocity is increasing at $t = 2$.

h) Find the velocity and position when acceleration is zero.

$0 = 4t - 4$ $v(1) = 2(1)^2 - 4(1) = -2$

$1 = t$ $s(1) = -7/3$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $f'(-1) = \frac{f(-1+h) - f(-1)}{h}$

7. Find $\lim_{h\to 0} \frac{2(-1+h)^3 - 3(-1+h) - 1}{h}$ Find $f'(-1)$ $f'(-1) = 6(-1)^2 - 3 = 6 - 3 = 3$

$f(x) = 2x^3 - 3x$ $f'(x) = 6x^2 - 3$ $\boxed{f'(-1) = 3}$

8. If $f(x) = x^4 - 4x$, evaluate $\lim_{x \to -1} \frac{f(x) - f(-1)}{x+1}$. Find $f'(-1)$

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ $f'(x) = 4x^3 - 4$ $f'(-1) = 4(-1)^3 - 4 = \boxed{-8}$

$\boxed{\text{a.) } -8}$ b.) 0 c.) 1 d.) 2 e.) 4

9. Find the points at which the graph of the equation has a vertical or horizontal tangent line:

$4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2y\left(\frac{dy}{dx}\right) - 8 + 4\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx}(2y + 4) = 8 - 8x$

$\frac{dy}{dx} = \frac{2(4 - 4x)}{2(y+2)} = \frac{4 - 4x}{y + 2}$

horizontal tangent

set numerator $f'(x) = 0$

$4 - 4x = 0$ $4 = 4x$

$x = 1$

$4(1)^2 + y^2 - 8(1) + 4y + \cancel{4} = 0$

$y^2 + 4y = 0$ $y(y+4) = 0$

$y = 0, -4$

$\boxed{(1,0) \text{ and } (1,-4)}$

vertical tangent

set denominator $f'(x) = 0$

$y + 2 = 0$ $y = -2$

$4x^2 + 4 - 8x - 8 + 4 = 0$

$4x^2 - 8x = 0$

$4x(x - 2) = 0$ $x = 0, 2$

$\boxed{(0,-2) \text{ and } (2,-2)}$

Chapter 3 Derivatives Test Review #2

33

1. Given $x^2y + y^2 = 2x$
   a. Find dy/dx

   b. Find a point on the graph where there is a horizontal tangent

   c. Find the points on the graph where there is a vertical tangent

2. Find d^2y/dx^2 for $\frac{x+2}{3-x}$

3. Find dy/dx for $\left(\frac{3x+1}{1-x^2}\right)^3$

(34)

Chapter 2 Derivatives Test Review #2

Key

1. Given $x^2y + y^2 = 2x$ (f, g)

a. Find dy/dx

$$2xy + x^2\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 2$$

$$x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2 - 2xy$$

$$\frac{dy}{dx}\left(x^2 + 2y\right) = 2 - 2xy$$

$$\boxed{\frac{dy}{dx} = \frac{2-2xy}{x^2+2y}}$$

b. Find the points on the graph where there is a horizontal tangent

*set numerator of derivative = 0

$$2 - 2xy = 0$$
$$2 = 2xy$$
$$xy = 1$$
$$x = 1,\ y = 1$$

*check with equation $x^2y + y^2 = 2x$

$(1)^2(1) + (1)^2 = 2(1)$ ✓

$$\boxed{(1,1)}$$

c. Find the points on the graph where there is a vertical tangent

*set denominator of derivative = 0

$$x^2 + 2y = 0$$
$$0^2 + 2(0) = 0$$
$$(-2)^2 + 2(-2) = 0$$

*check with equation

$0^2(0) + 0^2 = 2(0)$ ✓

$(-2)^2(-2) + (-2)^2 = 2(-2)$ ✓

$\boxed{(0,0) \text{ and } (-2,-2)}$

2. Find d^2y/dx^2 for $\frac{x+2}{3-x}$

$$\frac{f'g - fg'}{g^2}$$

$$\frac{dy}{dx} = \frac{(1)(3-x) - (x+2)(-1)}{(3-x)^2}$$

$$= \frac{3 - x + x + 2}{(3-x)^2} = \frac{5}{(3-x)^2}$$

$$\frac{dy}{dx} = 5(3-x)^{-2}$$

$$\frac{d^2y}{dx^2} = (-2)(5)(3-x)^{-3}(-1) = \boxed{\frac{10}{(3-x)^3}}$$

OR $$\frac{0(3-x)^2 - 5(2)(3-x)'(-1)}{(3-x)^4} = \boxed{\frac{10}{(3-x)^3}}$$

3. Find dy/dx for $\left(\frac{3x+1}{1-x^2}\right)^3$

*chain (first)

*quotient

$$\frac{dy}{dx} = 3\left(\frac{3x+1}{1-x^2}\right)^2\left[\frac{3(1-x^2) - (3x+1)(-2x)}{(1-x^2)^2}\right] = \frac{3(3x+1)^2(3x^2+2x+3)}{(1-x^2)^2(1-x^2)^2}$$

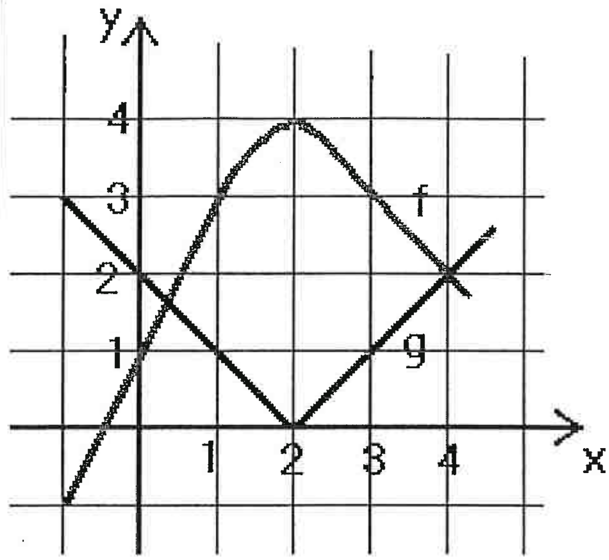
$$3 - 3x^2 + 6x^2 + 2x$$

$$\boxed{\frac{dy}{dx} = \frac{3(3x+1)^2(3x^2+2x+3)}{(1-x^2)^4}}$$

35

1. Let h(t) = f [g(t)]. Find h'(1)

2. Let z(t) = f(t)g(t). Find z'(3)



3.

Consider the curve defined by $2y^3 + 6x^2y - 12x^2 + 6y = 1$.

(a) Show that $\frac{dy}{dx} = \frac{4x - 2xy}{x^2 + y^2 + 1}$.

4. Find $\frac{d^2y}{dx^2}$ for $f(x) = \frac{x-1}{x+2}$

36

5. The position of the particle traveling along a straight line is $x(t) = t^3 - 9t^2 + 15t + 3$ for $t \geq 0$

a) Find when the particle is moving to the left.

b) Is speed increasing or decreasing at t = 2?

c) Find avg. acceleration in interval [0, 2]

d) Where is the particle located when the velocity is zero?

e) Find particle's displacement from t = 0 to t = 3

f) Find particle's distance from t = 0 to t = 3

g) Is the velocity increasing or decreasing at t=2?

h) Find the velocity and position when acceleration is zero.

6. Find $\frac{dy}{dx}$ for $f(x) = \sqrt{x}(x^3-7)^4$.

7. Find $\frac{dy}{dx}$ for $f(x) = \frac{\sqrt{2-x^2}}{4+3x}$.

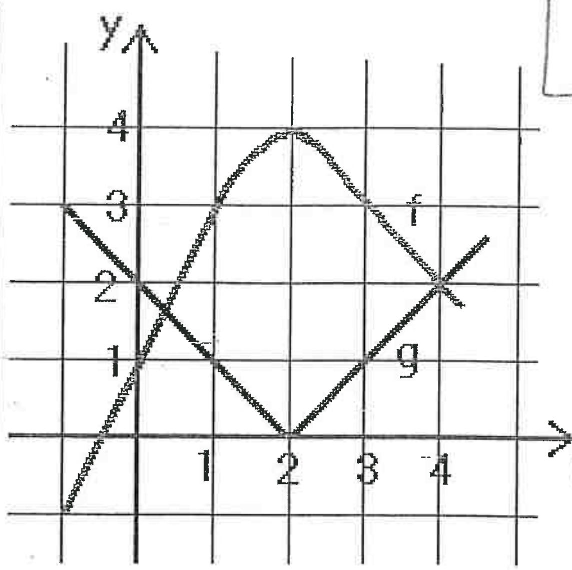
AB Calculus Ch. 2 Derivatives   Morning Test Review

Key (37)

*chain Rule $f'[g(t)]\cdot g'(t)$

1. Let h(t) = f [g(t)]. Find h'(1)

$$h'(t)=f'[g(t)]\cdot g'(t)$$
$$h'(1)=f'[g(1)]\cdot g'(1)$$
$$=f'[1]\cdot g'(1)$$
$$=(1)(-1)=\boxed{-1}$$



2. Let z(t) = f(t)g(t). Find z'(3)

$$z'(t)=f'(t)g(t)+f(t)g'(t)$$
$$z'(3)=f'(3)g(3)+f(3)g'(3)$$
$$=(-1)(1)+(3)(1)$$
$$=-1+3$$
$$=\boxed{2}$$

3.

*implicit, product Rule

Consider the curve defined by $2y^3+6x^2y-12x^2+6y=1$.

(a) Show that $\dfrac{dy}{dx}=\dfrac{4x-2xy}{x^2+y^2+1}$.

$$6y^2\frac{dy}{dx}+12xy+6x^2\left(\frac{dy}{dx}\right)-24x+6\frac{dy}{dx}=0$$
$$\frac{dy}{dx}(6y^2+6x^2+6)=24x-12xy$$
$$\frac{dy}{dx}=\frac{24x-12xy}{6y^2+6x^2+6}=\boxed{\frac{4x-2xy}{x^2+y^2+1}}$$

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4. Find $\frac{d^2y}{dx^2}$ for $f(x) = \frac{x-1}{x+2}$

$$\frac{dy}{dx} = \frac{(1)(x+2)-(x-1)(1)}{(x+2)^2}$$

$$= \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

$$\frac{dy}{dx} = 3(x+2)^{-2}$$

$$\frac{d^2y}{dx^2} = 3(-2)(x+2)^{-3}(1) = \boxed{\frac{-6}{(x+2)^3}}$$

$v(t) = 3t^2 - 18t + 15 \rightarrow 3(t^2-6t+5) \rightarrow 3(t-5)(t-1)$

$a(t) = 6t - 18$

5. The position of the particle traveling along a straight line is $x(t) = t^3 - 9t^2 + 15t + 3$ for $t \geq 0$

a) Find when the particle is moving to the left.

$+ \quad - \quad +$

$0 \quad 1 \quad 5$

$\boxed{(1,5) \text{ since } v(t) < 0}$

b) Is speed increasing or decreasing at t = 2?

Since $v(2) < 0$ and $a(2) < 0$ have same signs, speed increasing at $t = 2$.

c) Find avg. acceleration in in interval [0, 2]

$= \frac{v(2)-v(0)}{2-0} = \frac{-9-15}{2-0} = \frac{-24}{2} = \boxed{-12 \text{ units}/s^2}$

$v(2) = -9$

$v(0) = 15$

d) Where is the particle located when the velocity is zero?

$v(t) = 0$ at $t = 1, 5$

$\boxed{s(1) = 10 \quad s(5) = -22}$

e) Find particle's displacement from t = 0 to t = 3

$s(3) = -6$

$s(0) = 3$

$s(3) - s(0)$

$-6 - 3 = \boxed{-9}$

f) Find particle's distance from t = 0 to t = 3

$s(0) = 3$

> 7

$s(1) = 10$

> 16

$s(3) = -6$

$7 + 16 = \boxed{23}$

g) Is the velocity increasing or decreasing at t=2?

*Is $a(t) > 0$ or $a(t) < 0$

$a(2) = -6$, so since $a(2) < 0$, velocity decreasing at $t = 2$.

h) Find the velocity and position when acceleration is zero.

$a(t) = 6t - 18$

$0 = 6(t-3)$

$t = 3$

$\boxed{s(3) = -6 \quad v(3) = -12}$

6. Find $\frac{dy}{dx}$ for $f(x) = \sqrt{x}(x^3-7)^4$ $\quad (x)^{1/2}(x^3-7)^4$

product, chain (first)

$f'g + fg'$

$$\frac{1}{2}(x)^{-1/2}(x^3-7)^4 + x^{1/2} \cdot 4(x^3-7)^3(3x^2)$$

$$= \boxed{\frac{(x^3-7)^4}{2\sqrt{x}} + 12x^{5/2}(x^3-7)^3}$$

7. Find $\frac{dy}{dx}$ for $f(x) = \frac{\sqrt{2-x^2}}{4+3x} = \frac{(2-x^2)^{1/2}}{4+3x}$

* quotient (first)
* chain

$$f'(x) = \frac{\frac{1}{2}(2-x^2)^{-1/2}(-2x)(4+3x) - (2-x^2)^{1/2}(3)}{(4+3x)^2}$$

$$f'(x) = \frac{\frac{-x(4+3x)}{\sqrt{2-x^2}} - 3\sqrt{2-x^2}}{(4+3x)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{-4x - 3x^2 - 3(2-x^2)}{(4+3x)^2(2-x^2)^{1/2}} = \frac{-4x-3x^2-6+3x^2}{(4+3x)^2(2-x^2)^{1/2}}$$

$$= \boxed{\frac{-4x-6}{(4+3x)^2(2-x^2)^{1/2}}}$$

A.P. Calculus AB Ch.3 Review Session Problems (WS #4)

1) Consider the curve given by $x^2 - xy + y^2 = 4$.

a) Find the two points on the curve at $x = 0$

b) Find $\frac{dy}{dx}$ by differentiating implicitly.

c) Use $\frac{dy}{dx}$ to find the slope of the lines tangent to the curve at the points found in part **a**.

d) Write the equations of the line tangent to the curve at the points above.

e) Set up and write the equations (don't solve) in order to find vertical and horizontal tangents to this graph curve.

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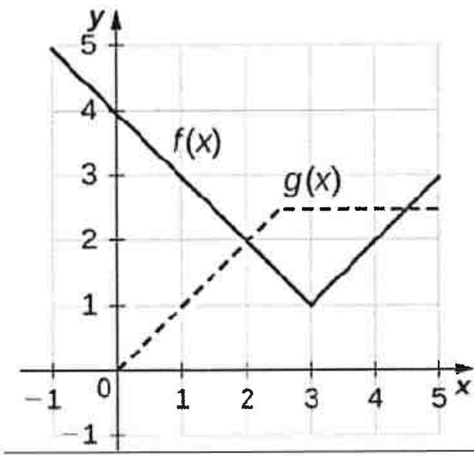
2) If g is differentiable everywhere and $g(x) = \begin{cases} 6x^3 - 8x^2 + 8, & x < -2 \\ ax + b, & x \geq -2 \end{cases}$, find a and b

(Involve derivatives in your work)

3)



a) If $h(x) = f(x) \cdot g(x)$, find $h'(2)$

b) If $w(x) = f(g(x))$, find $w'(3)$

c) $If\ z(x) = 2[f(x)]^3 \quad find\ z'(0)$

d) If $k(x) = f(f(x))$, find $k'(4)$

e) If $p(x) = \frac{g(x)}{f(x)}$ find $p'(1)$

4) Find $\frac{dy}{dx}$ for $y = 2\left(\frac{5x^3-2}{3-x^2}\right)^7$ (Write derivative as a simplified rational expression)

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5) A particle moves along a straight line according to the given equation: x(t) = $\frac{t^4}{4} - t^3 - 2t^2 + 1$, for all real numbers in **meters per minute**

a) Find the velocity and acceleration function

b) Find when the particle changes direction (justify with because statement)

c) Determine interval when particle is moving left (justify with because statement)

d) Determine interval when particle is moving left (justify with because statement)

e) Find the average velocity of particle in interval [1, 3] (show work, include units)

f) Find the average acceleration of particle in interval [1, 3] (show work, include units)

g) Find particle's displacement from t = 1 to t = 6 (Show your work)

h) Find particle's distance from t = 1 to t = 6 (Show your work)

i) At t = 2, is the speed increasing or decreasing? Provide justification for your answer.

j) At t = 5, is the velocity increasing or decreasing? Provide justification for your answer.

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Key

A.P. Calculus AB   2.2-2.5 Review Session Problems (WS #4)

1) Consider the curve given by $x^2 - xy + y^2 = 4$.

a) Find the two points on the curve at $x = 0$

$(0)^2 - (0)y + y^2 = 4$

$y^2 = 4$

$\sqrt{y^2} = \pm\sqrt{4}$

$y = 2, -2$

$\boxed{\text{The 2 points are } (0,2) \text{ and } (0,-2)}$

b) Find $\frac{dy}{dx}$ by differentiating implicitly.

* product rule

$x^2 - (\overbrace{xy}^{f\,g}) + y^2 = 4$

$2x - \left(\underbrace{(1)}_{f'}\underbrace{(y)}_{g} + \underbrace{(x)}_{f}\underbrace{(1)\left(\frac{dy}{dx}\right)}_{g'}\right) + 2y\left(\frac{dy}{dx}\right) = 0$

$2x - y - x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$

$-x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = -2x + y$

$\frac{dy}{dx}(-x + 2y) = -2x + y$

$$\boxed{\frac{dy}{dx} = \frac{-2x+y}{-x+2y}}$$

c) Use $\frac{dy}{dx}$ to find the slope of the lines tangent to the curve at the points found in part a.

i) point: $(0,2)$

slope: $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{-2(0)+2}{-0+2(2)} = \frac{2}{4}$

$\boxed{\text{slope: } m = 1/2}$

$\boxed{y - 2 = \frac{1}{2}(x - 0)}$

ii) point: $(0,-2)$

slope: $\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{-2(0)-2}{-0+2(-2)} = \frac{-2}{-4}$

$\boxed{m = 1/2}$

$\boxed{y + 2 = \frac{1}{2}(x - 0)}$

d) Write the equation of the line tangent to the curve at the points above

$\boxed{y - 2 = \frac{1}{2}(x - 0)}$ $\boxed{y + 2 = \frac{1}{2}(x - 0)}$

e) Detail how you would set up (don't solve) in order to find vertical and horizontal tangent:

horizontal tangent: set numerator of $\frac{dy}{dx} = 0 \rightarrow \boxed{-2x + y = 0}$

vertical tangent: set denominator of $\frac{dy}{dx} = 0 \rightarrow \boxed{-x + 2y = 0}$

2) If g is differentiable everywhere and $g(x)=\begin{cases}6x^3-8x^2+8, & x<-2\\ ax+b, & x\ge -2\end{cases}$, find a and b

(Involve derivatives in your work)

continuous property: set equations equal (at $x=-2$)

$6x^3-8x^2+8=ax+b$

$6(-2)^3-8(-2)^2+8=a(-2)+b$

$-72=-2a+b$

$-72=-2(104)+b$

$\boxed{136=b}$

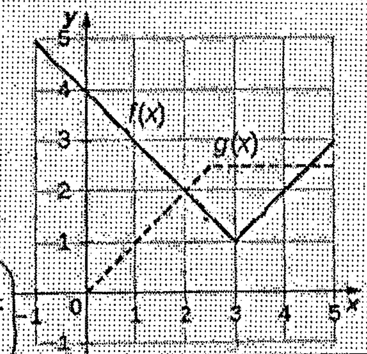
differentiable property: set derivatives equal (at $x=-2$)

$18x^2-16x=a$

$18(-2)^2-16(-2)=a$

$\boxed{a=104}$

3)



a) If $h(x)=f(x)\cdot g(x)$, find $h'(2)$ *product rule

$h'(x)=f'(x)g(x)+f(x)g'(x)$

$h'(2)=f'(2)g(2)+f(2)g'(2)$

$h'(2)=(-1)(2)+(2)(1)$

$\boxed{h'(2)=0}$

b) If $w(x)=f(g(x))$, find $w'(3)$ *chain rule

$w'(x)=f'[g(x)]\cdot g'(x)$

$w'(3)=f'[g(3)]\cdot g'(3)$

$w'(3)=f'[2.5]\cdot g'(3)$

$w'(3)=(-1)(0)$

$\boxed{w'(3)=0}$

*chain rule

out: $2[\ \ ]^3$

in: $f(x)$

c) If $z(x)=2[f(x)]^3$ find $z'(0)$

$z'(x)=6(\ \ )^2\cdot f'(x)$

$z'(x)=6[f(x)]^2\cdot f'(x)$

$z'(0)=6[f(0)]^2\cdot f'(0)$

$z'(0)=6[4]^2\cdot(-1)$

$\boxed{z'(0)=-96}$

d) If $k(x)=f(f(x))$, find $k'(4)$ ← chain rule

$k'(x)=f'[f(x)]\cdot f'(x)$

$k'(4)=f'[f(4)]\cdot f'(4)$

$k'(4)=f'[2]\cdot f'(4)$

$k'(4)=(-1)(1)$

$\boxed{k'(4)=-1}$

e) If $p(x)=\frac{g(x)}{f(x)}$ find $p'(1)$ *quotient rule

$$p'(x)=\frac{g'(x)f(x)-g(x)f'(x)}{[f(x)]^2}$$

$$p'(1)=\frac{g'(1)f(1)-g(1)f'(1)}{[f(1)]^2}$$

$$p'(1)=\frac{(1)(3)-(1)(-1)}{3^2}$$

$$p'(1)=\frac{3+1}{9}=\boxed{\frac{4}{9}}$$

4) Find $\frac{dy}{dx}$ for $y=2\left(\frac{5x^3-2}{3-x^2}\right)^7$ (Write derivative as a simplified rational expression)

$45x^2-15x^4+10x^4-4x$

*i) chain rule

ii) quotient rule

out: $2[\ \ ]^7$

in: $\frac{5x^3-2}{3-x^2}$ ← quotient rule

$$y'=14\left[\frac{5x^3-2}{3-x^2}\right]^6\cdot\left[\frac{\overbrace{15x^2}^{f'}\overbrace{(3-x^2)}^{g}-\overbrace{(5x^3-2)}^{f}\overbrace{(-2x)}^{g'}}{(3-x^2)^2}\right]$$

$$y'=14\left[\frac{5x^3-2}{3-x^2}\right]^6\left[\frac{-5x^4+45x^2-4x}{(3-x^2)^2}\right] \quad g^2$$

$$\boxed{y'=\frac{14(5x^3-2)^6(-5x^4+45x^2-4x)}{(3-x^2)^8}}$$

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5) A particle moves along a straight line according to the given equation: $x(t) = \frac{t^4}{4} - t^3 - 2t^2 + 1$, for all real numbers in **meters per minute**

a) Find the velocity and acceleration function

$v(t) = t^3 - 3t^2 - 4t$

$a(t) = 3t^2 - 6t - 4$

b.) Find when the particle changes direction (justify with because statement)

$0 = t(t^2 - 3t - 4)$

$0 = t(t-4)(t+1)$

$t = 0, -1, 4$

v(t): − + − + at −2, −1/2, 1, 5; critical values −1, 0, 4

Direction change at $t = -1, 0, 4$ b/c $v(t)$ change signs

c) Determine interval when particle is moving left (justify with because statement)

particle moves left $(-\infty, -1) \cup (0, 4)$ b/c $v(t) < 0$

d) Determine interval when particle is moving left (justify with because statement)

particle moves right $(-1, 0) \cup (4, \infty)$ b/c $v(t) > 0$

e) Find the average velocity of particle in interval [1, 3] (show work, include units)

Avg. velocity $= \frac{\text{change in position}}{\text{change in time}} \rightarrow \frac{x(3) - x(1)}{3-1}$

$x(3) = -23.75$

$x(1) = -1.75$

$\frac{-23.75 - (-1.75)}{3-1} = \boxed{-11 \text{ meters/min}}$

f) Find the average acceleration of particle in interval [1, 3] (show work, include units)

Avg. acceleration $= \frac{\text{change in velocity}}{\text{change in time}} \rightarrow \frac{v(3) - v(1)}{3-1}$

$v(3) = -12$

$v(1) = -6$

$\frac{-12 - (-6)}{3-1} = \boxed{-3 \text{ meters/min}^2}$

g) Find particle's displacement from t = 1 to t = 6 (Show your work)

displacement = final position − initial position

displacement $= x(6) - x(1)$

$= 37 - (-1.75)$

$x(6) = 37$

$x(1) = -1.75$

$\boxed{\text{displacement is } 38.75\text{ m}}$

h) Find particle's distance from t = 1 to t = 6 (Show your work) *count the distance of endpoints to locations of direction change

(*direction change at $t=4$)

$x(1) = -1.75$

$x(4) = -31$

$x(6) = 37$

x(t): −31, −1.75, 37; 29.25; 68

$\boxed{\text{Total distance is } 29.25 + 68 = 97.25\text{ m}}$

i) At t = 2, is the speed increasing or decreasing? Provide justification for your answer.

$v(2) = -12 \quad a(2) = -4$

$\boxed{\text{Speed is increasing since } v(t) \text{ and } a(t) \text{ have same signs at } t=2}$

j) At t = 5, is the velocity increasing or decreasing? Provide justification for your answer.

$a(5) = 41$ *This phrasing is describing acceleration, not velocity

$\boxed{\text{Velocity is increasing at } t=5 \text{ since acceleration is positive}}$

Ch. 2-3 Test Topics:

1) Extended Implicit Differentiation Problem.
Implicit Differentiation FRQ problem

2) Graph Problems Miscellaneous Ch 3 Review WS.

3) Derivative problems involving
(Power/Product/Quotient/Chain/Implicit)

4) PVA Particle Motion Problem (similar to one on Ch. 2 quiz)

5) Differentiable Piecewise Problem
(Differentiable Piecewise Function WS)