Solutions:

7. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

(A) 5 (B) 6 (C) 8 (D) 10 (E) 12

7. **Answer (C):** Let a be the initial term and d the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is

$$a + (a + d) + \cdots + (a + 11d) = 12a + 66d = 360,$$

so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for a, namely $a = 8$.

8. An *iterative average* of the numbers 1, 2, 3, 4, and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

(A) $\frac{31}{16}$ (B) 2 (C) $\frac{17}{8}$ (D) 3 (E) $\frac{65}{16}$

8. **Answer (C):** If the numbers are arranged in the order a, b, c, d, e, then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a+b+2c+4d+8e}{16}.$$

The largest value is obtained by letting $(a, b, c, d, e) = (1, 2, 3, 4, 5)$ or $(2, 1, 3, 4, 5)$, and the smallest value is obtained by letting $(a, b, c, d, e) = (5, 4, 3, 2, 1)$ or $(4, 5, 3, 2, 1)$. In the former case the iterative average is $65/16$, and in the latter case the iterative average is $31/16$, so the desired difference is

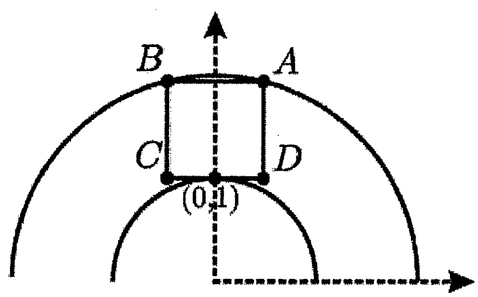
$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

12. A square region $ABCD$ is externally tangent to the circle with equation $x^2+y^2=1$ at the point $(0,1)$ on the side CD. Vertices A and B are on the circle with equation $x^2+y^2=4$. What is the side length of this square?

(A) $\frac{\sqrt{10}+5}{10}$ (B) $\frac{2\sqrt{5}}{5}$ (C) $\frac{2\sqrt{2}}{3}$ (D) $\frac{2\sqrt{19}-4}{5}$ (E) $\frac{9-\sqrt{17}}{5}$

12. **Answer (D):**

Suppose by symmetry that $A=(a,b)$ with $a>0$. Because $ABCD$ is tangent to the circle with equation $x^2+y^2=1$ at $(0,1)$ and both A and B are on the concentric circle with equation $x^2+y^2=4$, it follows that $B=(-a,b)$. Then the horizontal length of the square is $2a$ and its vertical height is $b-1$. Therefore $2a=b-1$, or $b=2a+1$. Substituting this into the equation $a^2+b^2=4$ leads to the equation $5a^2+4a-3=0$. By the quadratic formula, the positive root is $\frac{1}{5}(\sqrt{19}-2)$, and so the side length $2a$ is $\frac{1}{5}(2\sqrt{19}-4)$.



13. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 PM. How long, in minutes, was each day's lunch break?

(A) 30 (B) 36 (C) 42 (D) 48 (E) 60

13. **Answer (D):** Let the length of the lunch break be m minutes. Then the three painters each worked $480-m$ minutes on Monday, the two helpers worked $372-m$ minutes on Tuesday, and Paula worked $672-m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$(p+h)(480-m)=50,$$
$$h(372-m)=24, \text{ and}$$

13 (continued)

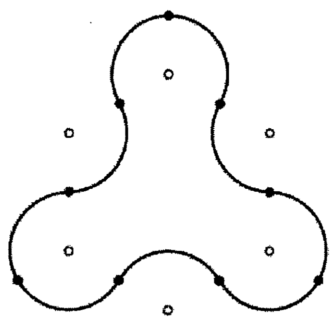
$$p(672 - m) = 26.$$

Adding the last two equations gives $672p + 372h - mp - mh = 50$, and subtracting this equation from the first one gives $108h - 192p = 0$, so $h = \frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\frac{25p}{9}(480 - m) = 50,$$
$$p(672 - m) = 26.$$

The solution of this system is $p = \frac{1}{24}$ and $m = 48$. Note that $h = \frac{2}{27}$.

14. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?

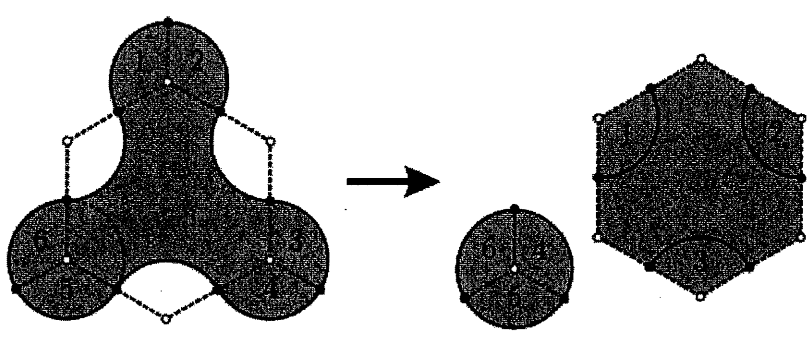


(A) $2\pi + 6$ (B) $2\pi + 4\sqrt{3}$ (C) $3\pi + 4$ (D) $2\pi + 3\sqrt{3} + 2$

(E) $\pi + 6\sqrt{3}$

14. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4} \cdot 2^2\right) = \pi + 6\sqrt{3}.$$



15. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

(A) $\frac{49}{512}$ (B) $\frac{7}{64}$ (C) $\frac{121}{1024}$ (D) $\frac{81}{512}$ (E) $\frac{9}{32}$

15. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of $BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

AMC 12A 2012 Practice Problems

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$a + (a+d) + \ldots + (a+11d) = 12a + 66d = 360$

$2a + 11d = 60$

$d = 4 \rightarrow 2a = 16$, $\boxed{a = 8^\circ}$

8. An *iterative average* of the numbers 1, 2, 3, 4, and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

(A) $\frac{31}{16}$ (B) 2 (C) $\frac{17}{8}$ (D) 3 (E) $\frac{65}{16}$

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a}{16}+\frac{b}{16}+\frac{c}{8}+\frac{d}{4}+\frac{e}{2} = \frac{a+b+2c+4d+8e}{16}$$

largest: $(1,2,3,4,5) \rightarrow \frac{65}{16}$

smallest: $(5,4,3,2,1) \rightarrow \frac{31}{16}$

$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \boxed{\frac{17}{8}}$

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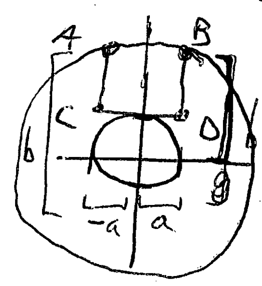

Square: side $2a$, height $b-1$

$2a = b-1$

$b = 2a+1$

$a^2+b^2 = 4$

$a^2+b^2=4$

$(a^2)+(2a+1)^2 = 4$

$a^2+4a^2+4a+1 = 4$

$5a^2+4a-3=0$

$\frac{-4 \pm \sqrt{16-4(5)(-3)}}{2(5)}$

$\frac{-4\pm\sqrt{76}}{10} = \frac{-4\pm 2\sqrt{19}}{10}$

$a = -\frac{2}{5} + \frac{\sqrt{19}}{5}$

$2a = 2\left(\frac{-2}{5}+\frac{\sqrt{19}}{5}\right) = \frac{1}{5}\left[-4+\sqrt{19}\right]$

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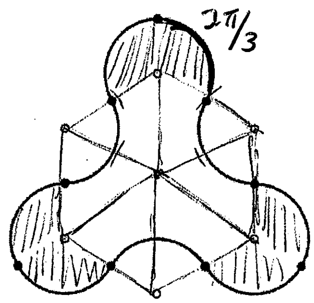

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