"AP Live" Calculus AB Cumulative Course Review WS

Key

Unit 1: Limits

1) Use Continuity Conditions to answer and justify the below question:

Is $f(x) = \begin{cases} \cos x, x < 0 \\ x^2+1, x \geq 0 \end{cases}$ continuous at $x = 0$?

i) $f(0) = 0^2+1 = 1$ ✓

ii) $\lim_{x\to0^-} \cos x = \cos 0 = 1$, $\lim_{x\to0^+} x^2+1 = 1$, $\lim_{x\to0} f(x) = 1$ ✓

iii) $f(0) = \lim_{x\to0} f(x) = 1$ ✓

$\boxed{\text{By continuity conditions } f(x) \text{ is continuous at } x=0}$

continuity conditions:

i) point exists: $f(c)$ exists

ii) limit exists $\lim_{x\to c} f(x)$ exists

$\left[\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\right]$

iii) $f(c) = \lim_{x\to c} f(x)$

2) **The function f is continuous at $x = 1$.**

If $f(x) = \begin{cases} \dfrac{\sqrt{x+3}-\sqrt{3x+1}}{x-1} & \text{for } x \neq 1 \\ k & \text{for } x = 1 \end{cases}$ then $k =$

*step through continuity conditions:

i) $f(1) = k$

ii) $\lim_{x\to1} \frac{\sqrt{x+3}-\sqrt{3x+1}}{x-1} = \frac{\sqrt4-\sqrt4}{1-1} = \frac{0}{0} \to \lim_{x\to1} \frac{(\sqrt{x+3}-\sqrt{3x+1})}{(x-1)} \cdot \frac{(\sqrt{x+3}+\sqrt{3x+1})}{(\sqrt{x+3}+\sqrt{3x+1})}$

conjugate method

$\lim_{x\to1} \frac{x+3-(3x+1)}{(x-1)(\sqrt{x+3}+\sqrt{3x+1})} = \overset{-2x+2}{x+3-3x-1}$

$\lim_{x\to1} \frac{-2\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x+3}+\sqrt{3x+1})} = \frac{-2}{\sqrt4+\sqrt4}$

$= \frac{-2}{4} = \frac{-1}{2}$

iii) $f(1) = \lim_{x\to1} f(x)$

$\boxed{K = -1/2}$

Unit 2-3: Derivatives / Derivatives of Composites

3) Given $h(x) = g(f(x))$

$h'(2) =$

*Recall chain rule:

$\frac{d}{dx} f[g(x)] = f'[g(x)] \cdot g'(x)$

$h(x) = g[f(x)]$

$h'(x) = g'[f(x)] \cdot f'(x)$

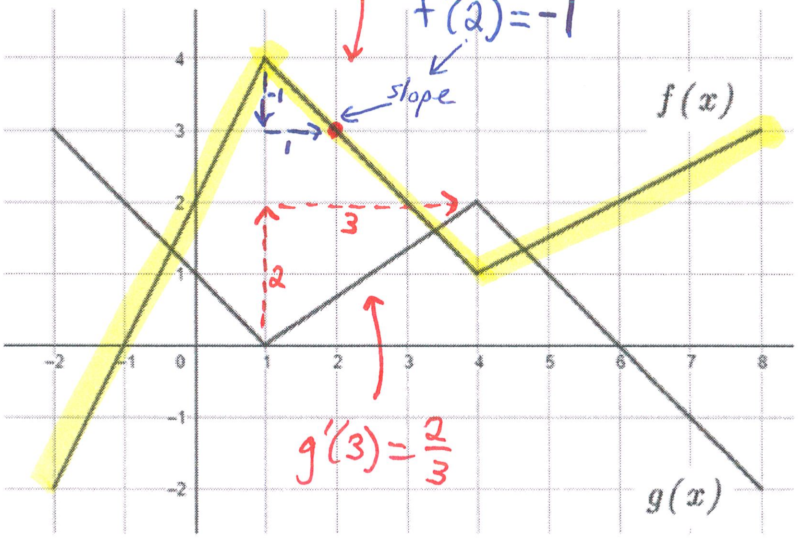
$h'(2) = g'[\underline{f(2)}] \cdot f'(2)$

$h'(2) = g'[\underline{3}] \cdot f'(2)$

$h'(2) = \frac{2}{3} \cdot (-1) \rightarrow \boxed{h'(2) = \frac{-2}{3}}$

$f(2) = 3$

$f'(2) = -1$



$g'(3) = \frac{2}{3}$

4) Find $h'(1)$ given $h(x)$

$$h(x) = \frac{\overbrace{k(x)}^{f}}{\underbrace{3x}_{g}}$$

x	-1	1
$k(x)$	-3	2
$k'(x)$	4	-5

*Recall quotient rule $\frac{f'g-fg'}{g^2}$

$$h'(x) = \frac{\overset{f'}{k'(x)}\cdot\overset{g}{3x} - \overset{f}{k(x)}\cdot\overset{g'}{3}}{\underbrace{[3x]^2}_{g^2}}$$

$$h'(1) = \frac{k'(1)\cdot 3(1) - k(1)\cdot 3}{[3(1)]^2}$$

$$h'(1) = \frac{(-5)(3)-(2)(3)}{3^2}$$

$$h'(1) = \frac{-15-6}{9} = \frac{-21}{9}$$

$$\boxed{h'(1) = -\frac{7}{3}}$$

5)

$$f(x) = \tan^2(3x^2)\,,\ f'(x) = ?$$

* Rewrite trig equation
* Apply nested chain rule

$$\frac{d}{dx}f[g(h(x))] = f'[g(h(x))]\cdot g'(h(x))\cdot h'(x)$$

$$f(x) = [\tan(3x^2)]^2$$

out: $[\ ]^2$
in: $\tan(\ )$
inner: $3x^2$

$$f'(x) = 2[\quad]\cdot\sec^2(\quad)\cdot 6x$$

$$f'(x) = 2[\tan(3x^2)]\sec^2(3x^2)\cdot 6x$$

$$\boxed{f'(x) = 12x\tan(3x^2)\sec^2(3x^2)}$$

6)

Given that $p(x) = \sqrt[3]{2x-1}$ find $[p^{-1}]'(5)$.

$$\sqrt[3]{2x-1} = 5$$

$$\left(\sqrt[3]{2x-1}\right)^3 = (5)^3$$

$$2x-1 = 125$$
$$2x = 126$$
$$x = 63$$

* Recall that function and its inverse at their corresponding points have slopes that are reciprocals of each other.

$p(63) = 5$	$(p^{-1})(5) = \underline{63}$
$p'(63) = n$	$(p^{-1})'(5) = \frac{1}{n}$

$$p(x) = (2x-1)^{1/3}$$

$$p'(x) = \frac{1}{3}(2x-1)^{-2/3}(2)$$

$$p'(x) = \frac{2}{3(2x-1)^{2/3}}$$

$$p'(63) = \frac{2}{3(2(63)-1)^{2/3}}$$

$$p'(63) = \frac{2}{3(25)} = \frac{2}{75}$$

$$\boxed{(p^{-1})'(5) = \frac{75}{2}}$$

7) If $y^2-3x=7$, then find $\frac{d^2y}{dx^2}$ in terms of x and y

$$2y\left(\frac{dy}{dx}\right) - 3 = 0$$

$$2y\left(\frac{dy}{dx}\right) = 3$$

$$\frac{dy}{dx} = \frac{3}{2y}$$

$$\frac{dy}{dx} = \frac{3}{2}y^{-1}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}\cdot -1y^{-2}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-3}{2y^2}\left(\frac{dy}{dx}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-3}{2y^2}\left(\frac{3}{2y}\right)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-9}{4y^3}}$$

8) The function f is defined on all the reals such that $f(x) = \begin{cases} x^2 + kx - 3 & for\ x \le 1 \\ 3x + b & for\ x > 1 \end{cases}$

For which of the following values of k and b will the function f be both continuous and differentiable on its entire domain?

(set derivatives equal to each other at x = 1)

$$f'(x) = \begin{cases} 2x+k & \text{for } x \le 1 \\ 3 & \text{for } x > 1 \end{cases}$$

Set equations equal: replace x=1

$$x^2+kx-3=3x+b$$

$$1^2+k(1)-3=3(1)+b$$

$$k-2=3+b \leftarrow$$

$$(1)-2=3+b$$

$$\boxed{-4=b}$$

(set equations equal at x=1)

Set derivatives equal (at x=1)

$$2x+k=3 \leftarrow \text{replace } x=1$$

$$2(1)+k=3$$

$$\boxed{k=1}$$

9) $If\ y = e^{kx}, then\ \frac{d^5y}{dx^5} =$

*find the 5th derivative *Recall that $\frac{d}{dx}e^u = e^u \cdot u'$

$$y=e^{kx}$$

$$y'=e^{kx}\cdot k = ke^{kx}$$

$$y''=ke^{kx}\cdot k = k^2e^{kx}$$

$$y'''=k^3e^{kx}$$

$$y''''=k^4e^{kx}$$

$$\boxed{\frac{d^5y}{dx^5}=k^5e^{kx}}$$

10) Consider the function $f(x) = \frac{\overbrace{6x}^{f}}{\underbrace{a+x^3}_{g}}$ $for\ which\ f'(0) = 3$

Find the value of a.

*Apply quotient rule: $\frac{f'g-fg'}{g^2}$

$$f'(x) = \frac{\overbrace{6}^{f'}\cdot\overbrace{(a+x^3)}^{g} - \overbrace{6x}^{f}\cdot\overbrace{3x^2}^{g'}}{\underbrace{(a+x^3)^2}_{g^2}}$$

$$f'(0)=\frac{6(a+0^3)-6(0)\cdot 3(0)^2}{(a+0^3)^2}$$

$$\downarrow$$

$$3=\frac{6a-0}{a^2}$$

$$\frac{3}{1}=\frac{6a}{a^2}$$

$$3a^2=6a$$

$$3a^2-6a=0$$

$$3a(a-2)=0 \nearrow \begin{array}{c|c} 3a=0 & a-2=0 \\ a=0 & a=2 \end{array}$$

$$\boxed{a=0,\ a=2}$$

Unit 4: Contextual Application of Differentiation

*Apply Related Rates

11) The positive variables **b** and **h** change with respect to time t. The relationship between b and h is given by the equation $h^3 = (4-2b)^2$. At the instant when $\frac{db}{dt} = 3$ and h = 4, what is the value of $\frac{dh}{dt}$?

plug in h=4 to find b: $4^3=(4-2b)^2$

$\sqrt{64}=\sqrt{(4-2b)^2}$

$\pm 8 = 4-2b$

$8=4-2b$ | $-8=4-2b$

$4=-2b$ | $-12=-2b$

$\cancel{-2=b}$ | $\boxed{6=b}$

$h^3=(4-2b)^2$

$3h^2\left(\frac{dh}{dt}\right)=2(4-2b)\left(-2\frac{db}{dt}\right)$

$3(4)^2\left(\frac{dh}{dt}\right)=2(4-2(6))(-2(3))$

$48\left(\frac{dh}{dt}\right)=2(-8)(-6)$

$48\left(\frac{dh}{dt}\right)=96$

$\boxed{\frac{dh}{dt}=2}$

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12) Determine $\frac{dz}{dt}$ if you know that $z = xy^2,\ z = 3,\ y = \frac{1}{2},\ \frac{dx}{dt} = -2,$ and $\frac{dy}{dt} = 5.$

*Apply Related Rates, product rule

plug in z,y to find x: $3=x\left(\frac12\right)^2 \rightarrow 3=x\left(\frac14\right)$

$12=x$

$z=\overbrace{x}^{f}\overbrace{y^2}^{g}$

$\frac{dz}{dt}=\overbrace{\frac{dx}{dt}}^{f'}\cdot\overbrace{y^2}^{g}+\overbrace{x}^{f}\cdot\overbrace{2y\left(\frac{dy}{dt}\right)}^{g'}$

$\frac{dz}{dt}=(-2)\left(\frac12\right)^2+(12)\cdot2\left(\frac12\right)(5)$

$\frac{dz}{dt}=\frac{-1}{2}+60$

$\boxed{\frac{dz}{dt}=59.5 \text{ or } \frac{119}{2}}$

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13) *The approximate value of* $y = \sqrt{3+e^x}$ at x = 0.08, obtained from the tangent line to the graph at x = 0 is

$y=(3+e^x)^{1/2}$

*Apply steps for linear approximation:

i) find ordered pair at x=0

ii) find slope → find y'(0)

iii) put in point-slope form and plug in decimal

$y(0)=\sqrt{3+e^0}=\sqrt4=2$

$y'(x)=\frac12(3+e^x)^{-1/2}(e^x)$

$y'(0)=\frac12(3+e^0)^{-1/2}(e^0)=\frac12\cdot\frac{1}{\sqrt4}=\frac14$

point: (0,2) slope: $m=\frac14$

$y-y_1=m(x-x_1)$

$y-2=\frac14(x-0)$

$y=\frac14(x)+2$

$y(0.08)=\frac14(0.08)+2$

$\boxed{y(0.08)=2.02}$

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14) ***The Function $C(x)$ gives the dollar cost of digging a hole x feet deep.***

$C(20) = 140$ means that a hole 20 ft deep costs \$140 to dig.

$C'(20) = 5$ means that when the hole is 20 feet deep, the cost of digging is increasing ✓ at a rate of \$5/foot.

set $f'(x)=0$

15) Let $f(x) = x^4 + ax^2 + b$. The graph of f has a relative maximum at $(0,1)$ and an inflection point when $x = 1$. The values of a and b are:

set $f''(x)=0$

plug in point (0,1), then plug in a=-6 to find b

$f'(x) = 4x^3 + 2ax + 0$

$0 = 4x^3 + 2ax$

$f''(x) = 12x^2 + 2a$

$0 = 12x^2 + 2a$

$0 = 12(1)^2 + 2a$

$-2a = 12$

$\boxed{a=-6}$

$f(x) = x^4 + ax^2 + b$

$f(0) = 0^4 + a(0)^2 + b$

$1 = 0 + (-6)(0) + b$

$\boxed{1 = b}$

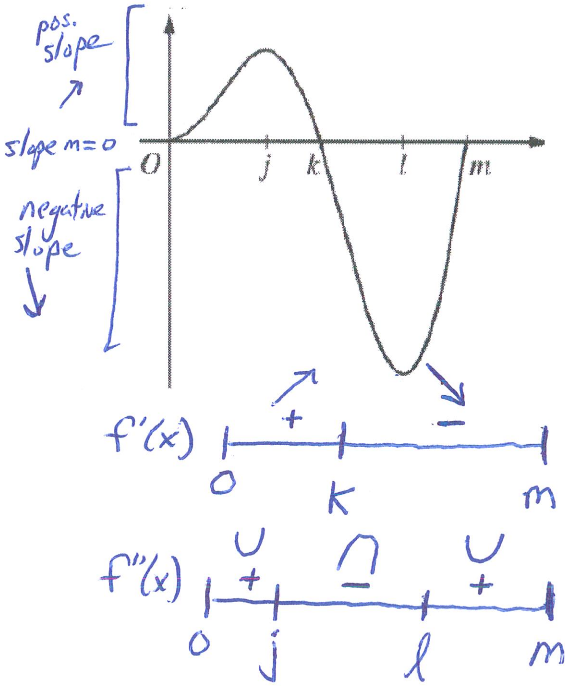
16) The f '(x) graph is shown . Answer the following:

a) Find the x-value where absolute minimum occurs

Absolute minimum occurs at x = m since graph falls more in interval k<x<m than it does rising 0<x<K

b) Find the x-value where the absolute maximum occurs

Absolute maximum occurs at x = K since f(x) graph is increasing 0<x<K and decreasing in interval K<x<m.

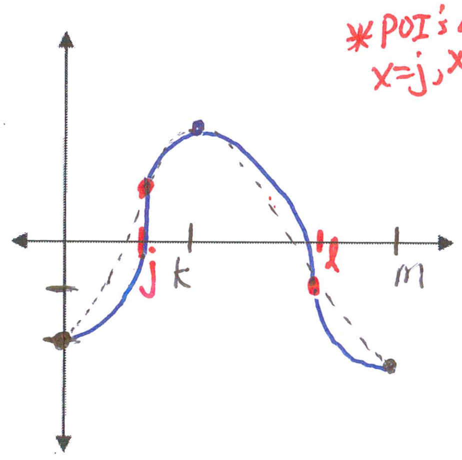


c) Sketch a possible f(x) graph given that f(0) = -2. (below)

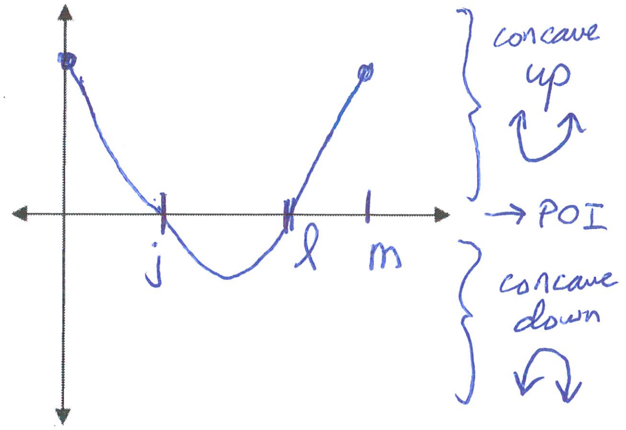
d) Sketch a possible f ''(x) graph (below)

c) $f(x)$

*POI's at x=j, x=l



d) $f''(x)$

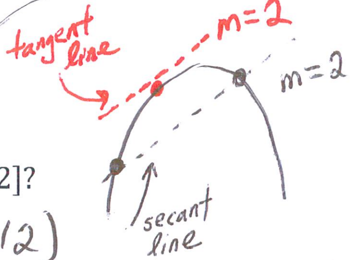


17) *The table below gives selected values for the differentiable function g.*

x	0	2	6	8	11	12
g(x)	-4	5	2	5	10	20

a) What's the least number of times g(x) = 3 in the given interval above? Justify your answer.

By IVT (Intermediate Value Theorem), since $g(x)$ is continuous on $[0, 12]$, $g(x)$ will reach value of $y=3$ at least 3 times.



b) What can be concluded with Mean Value Theorem on the interval [0, 12]?

By MVT, since $g(x)$ is continuous $[0,12]$ and differentiable $(0,12)$

then $g'(c) = \frac{g(12)-g(0)}{12-0}$ | $g'(c) = \frac{20-(-4)}{12}$

$g'(c) = \frac{24}{12} = 2$

Since slope of secant line at endpoints is 2, there must be at least one point on interval $(0,12)$ where slope of tangent line is 2.

c) Can Rolle's Theorem be applicable in the interval [2, 8]? Justify your answer.

By Rolle's Theorem, since $g(x)$ is continuous $[2,8]$ and differentiable $(2,8)$ and $g(2)=g(8)=5$, there must be $g'(c)=0$ on interval $(2,8)$

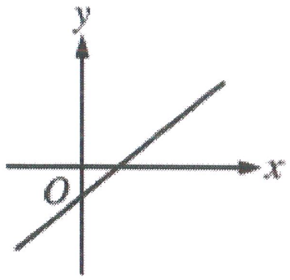
* since endpts have the same y-values, there must be slope of tangent line = 0.

18) *The function g is differentiable and increasing for all real numbers x, and the graph of f has exactly 2 points of inflections. Of the following, which could be the graph of g', the derivative of g?*

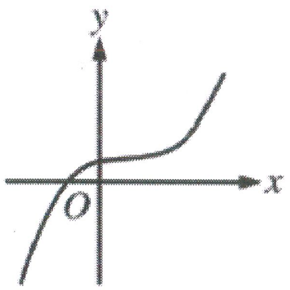
* since $g(x)$ is increasing, $g'(x)$ will have graph all above x-axis.

* points of inflections are represented on $g'(x)$ graph as peaks/valleys (max/mins)

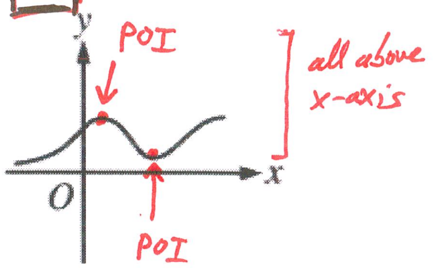
A)



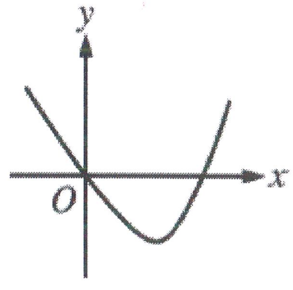
B)



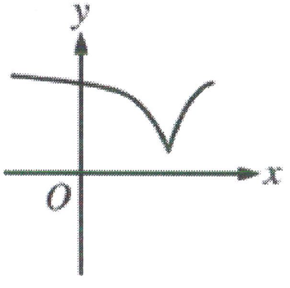
[✓ C)]



D)



E)



Unit 6: Integration and Accumulation of Change

19) Let f be the function defined by $f(x) = \int_0^x (2t^3 - 15t^2 + 36t)dt$. On which of the following interval is the graph of $y = f(x)$ concave down? → find 2nd derivative, set $f''(x)=0$, find critical point, create $f''(x)$ sign line.

$$f'(x) = \frac{d}{dx}\int_0^x 2t^3 - 15t^2 + 36t\,dt = 2x^3 - 15x^2 + 36x$$

$$f''(x) = 6x^2 - 30x + 36$$

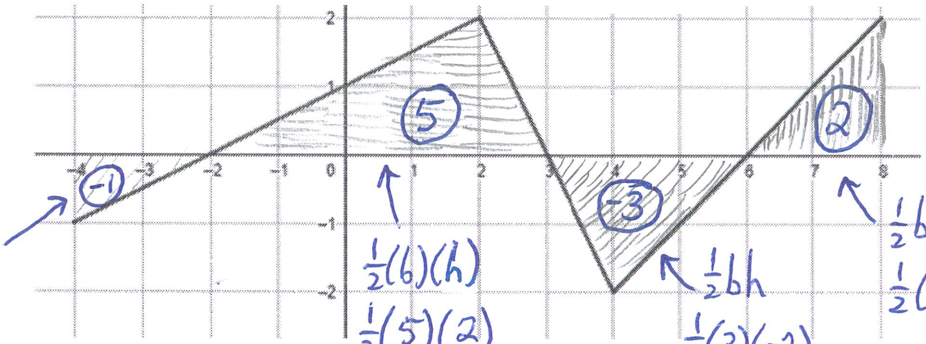
$$0 = 6(x^2 - 5x + 6)$$

$$0 = 6(x-2)(x-3)$$

$$x = 2,\ x = 3$$

$f''(x)$: $\overset{\cup}{+}$ | $\overset{\cap}{-}$ | $\overset{\cup}{+}$; 1, 2, 2.5, 3, 4

$\boxed{f(x) \text{ is concave down in } (2,3) \text{ since } f''(x) < 0}$



$\frac{1}{2}(b)(h)$
$\frac{1}{2}(2)(-1)$

$\frac{1}{2}(b)(h)$
$\frac{1}{2}(5)(2)$

$\frac{1}{2}bh$
$\frac{1}{2}(3)(-2)$

$\frac{1}{2}bh$
$\frac{1}{2}(2)(2)$

20) a) The graph of the piecwise linear function f is shown. What is the value of $\int_{-4}^{8} f(x)dx$?

* total the area regions b/t graph and x-axis.

$$\int_{-4}^{8} f(x)dx = -1 + 5 - 3 + 2 = \boxed{3}$$

b) The graph of the piecwise linear function f is shown. What is the value of $\int_{-4}^{8} f'(x)dx$? $f(8) = 2$, $f(-4) = -1$

* Recall FFTC:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$\int_{-4}^{8} f'(x)dx = f(8) - f(-4) = 2 - (-1) = \boxed{3}$$

c) The graph of the piecwise linear function f is shown. What is the value of $\int_{-3}^{7} f''(x)dx$? $f'(7) = 1$, $f'(-3) = \frac{1}{2}$ ← slope of graph at these points

$$\int_a^b f''(x)dx = f'(b) - f'(a)$$

$$\int_{-3}^{7} f''(x)dx = f'(7) - f'(-3) \rightarrow 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

21) Let f and g be continuous function such that $\int_0^8 f(x)dx = 12$, $\int_0^8 \frac{2g(x)dx}{2} = \frac{4}{2}$, $\rightarrow \int_0^8 g(x)dx = 2$

and $\int_5^8 (f(x) - g(x))dx = 3$. What is the value of $\int_0^5 (f(x) - g(x))dx$

$$\int_0^8 f(x)dx - \int_0^8 g(x)dx = 12 - 2 = \underline{\underline{10}}$$

$$\int_0^8 f(x) - g(x)dx = \int_0^5 f(x) - g(x)dx + \int_5^8 f(x) - g(x)dx$$

$$10 = \int_0^5 f(x) - g(x)dx + 3$$

$$7 = \int_0^5 f(x) - g(x)dx$$

$$\boxed{\int_0^5 (f(x) - g(x))dx = 7}$$

22) The function f is continuous and $\int_0^8 f(u)du = 6$. What is the value of $\int_1^3 xf(x^2-1)dx$?

*start with u-sub

$u = x^2 - 1$ | $dx = \frac{du}{2x}$

$\frac{du}{dx} = 2x$

$\int \cancel{x} \cdot f(u) \cdot \frac{du}{2\cancel{x}} \to \frac{1}{2}\int_0^8 f(u)du$

*convert bounds

if $x=1$, $u=x^2-1 \to u=0$

if $x=3$, $u=x^2-1 \to u=3^2-1=8$

$\frac{1}{2}\int_0^8 f(u)du = \frac{1}{2}(6)$

$= \boxed{3}$

23) If $\int_0^b (4bx - 2x^2)\,dx = 36$, then $b =$

$4b\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right)\Big]_0^b = 4b\left(\frac{b^2}{2}\right) - 2\left(\frac{b^3}{3}\right) - (0-0)$

$\frac{4b^3}{2} - \frac{2b^3}{3} = 36$

$\frac{4b^3}{3} = \frac{36}{1}$

$4b^3 = 108$

$b^3 = 27$

$\boxed{b=3}$

24) If $\int_{-2}^{2} (x^7 + k)dx = 16$, then $k =$

$\frac{x^8}{8} + kx\Big]_{-2}^{2} = \frac{2^8}{8} + k(2) - \left(\frac{(-2)^8}{8} + k(-2)\right)$

$\cancel{\frac{2^8}{8}} - \cancel{\frac{2^8}{8}} + 2k + 2k = 16$

$4k = 16$

$\boxed{k=4}$

25) **Unit 7: Differentiation Equations**

Consider the differential equation $\frac{dy}{dx} = \left(1 - \frac{4}{x^3}\right)(y-1)^2$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(2) = -1$. Find $f(1)$.

*cross multiply, separate variables.

$\frac{dy}{dx} = \frac{(1-\frac{4}{x^3})(y-1)^2}{1}$

$dy = (1-\frac{4}{x^3})(y-1)^2 dx$

$\int \frac{dy}{(y-1)^2} = \int \left(1 - \frac{4}{x^3}\right)dx$

$\int (\underline{y-1})^{-2} d\underline{y} = \int 1 - 4x^{-3}dx$

$u = y-1$

$\frac{du}{dy} = 1$

$dy = du$

$\int u^{-2}du = x - \frac{4x^{-2}}{-2} + C$

$\frac{u^{-1}}{-1}$

$\frac{-1}{(y-1)} = x + \frac{2}{x^2} + C$

plug in $(2,-1)$ to find C

$\frac{-1}{(-1-1)} = 2 + \frac{2}{2^2} + C$

$\frac{1}{2} = 2 + \frac{1}{2} + C$

$-2 = C$

$\frac{-1}{(y-1)^2} = x + \frac{2}{x^2} - 2$

$\frac{-1}{(y-1)^2} = \frac{x + \frac{2}{x^2} - 2}{1}$

$(y-1)^2\left(x + \frac{2}{x^2} - 2\right) = -1$

$(y-1)^2 = \frac{-1}{x^2 + \frac{2}{x^2} - 2}$

$\boxed{y = -\sqrt{\frac{-1}{x^2 + \frac{2}{x^2} - 2}} + 1}$

$y(1) = -\sqrt{\frac{-1}{1+2-2}} + 1 = -\sqrt{-1} + 1$

$\boxed{y(1) = \text{undefined}}$