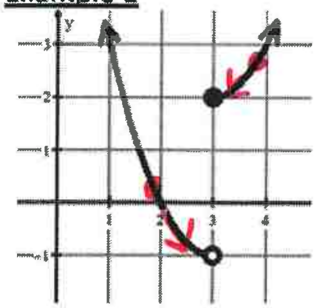
AP Calculus – 1.2 Notes – Techniques for finding Limits (Algebraic, Numeric, Graph)

Key

What is a *one-sided* limit?

A ***one-sided limit*** is the y-value a function approaches as you approach a given x-value from either the left or right side.

Example 1



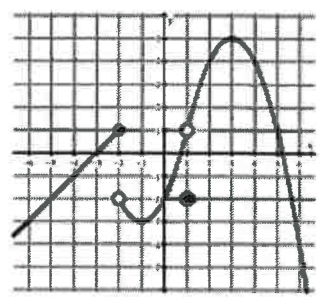
The limit of f as x approaches 3 from the left side is -1.	$\lim\limits_{x\to 3^-} f(x) = -1$
The limit of f as x approaches 3 from the right side is 2.	$\lim\limits_{x\to 3^+} f(x) = 2$

If the two sides are different? $\lim\limits_{x\to 3} f(x) =$ does not exist (dne)

b/c $\lim\limits_{x\to 3^-} f(x) \neq \lim\limits_{x\to 3^+} f(x)$ ← watch notation $\cancel{\lim\limits_{x\to 3^-} \neq \lim\limits_{x\to 3^+}}$

Example 2

a. $\lim\limits_{x\to -2^-} f(x) = 1$	b. $\lim\limits_{x\to -2^+} f(x) = -2$	c. $\lim\limits_{x\to -2} f(x) =$ dne
d. $\lim\limits_{x\to 1} f(x) = 1$	e. $\lim\limits_{x\to 0} f(x) = -2$	f. $\lim\limits_{x\to 3^-} f(x) = 5$
g. $\lim\limits_{x\to -1} f(x) = -3$	h. $f(1) = -2$	i. $f(-2) = 1$



Example 3

Sketch a graph of a function g that satisfies all of the following conditions.

a. $g(3) = -1$ → coordinate point $(3,-1)$

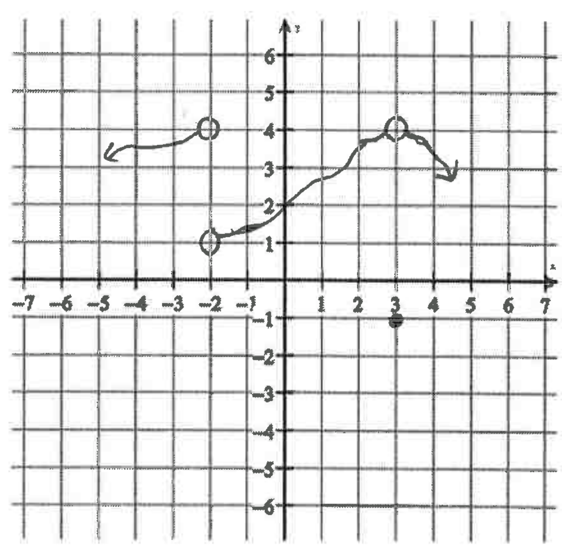
b. $\lim\limits_{x\to 3} g(x) = 4$

c. $\lim\limits_{x\to -2^+} g(x) = 1$

d. g is increasing on $-2 < x < 3$

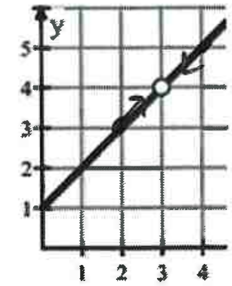
e. $\lim\limits_{x\to -2^-} g(x) > \lim\limits_{x\to -2^+} g(x)$

The graph is approaching a higher y-value from the left side of -2 than the y-value approach from the right side of $x=2$



Finding Limits from tables:

If we have the graph, it is easy to see the value of $\lim_{x\to3} f(x) = 4$



Without the graph, we could use a table of values.

x	2.9	2.99	3.01	3.1
$f(x)$	3.9	3.99	4.01	4.1

graph approaches y-value of 4 as x approaches 3

The function f is continuous and increasing for $x \geq 1$. The table gives values of f at selected values of x. Approximate the value of $\lim_{x\to2} \cos(f(x))$.

x	1.99	1.999	2.001	2.01
$f(x)$	4.85	4.999	5.001	5.15

$\cos(4.999) = 0.2827$

$\cos(5.001) = 0.2846$

$\lim_{x\to2} \cos(f(x)) =$ [illegible]283

Finding Limits Using Algebraic Methods:

*Indeterminate Form (limit does exist, some work required to find it)

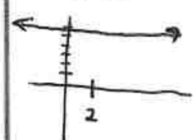
Direct Substitution

1. $\lim_{x\to-1} (x^2 + 2x - 4)$

$(-1)^2 + 2(-1) - 4$

$1 - 2 - 4 = \boxed{-5}$

2. $\lim_{x\to2} 6$



$\boxed{6}$

Factor and Cancel

3. $\lim_{x\to0} \frac{4x^2-5x}{x}$ $\frac{0}{0}$

$\lim_{x\to0} \frac{\cancel{x}(4x-5)}{\cancel{x}}$

$\lim_{x\to0} 4x-5 = \boxed{-5}$

4. $\lim_{x\to-7} \frac{2x^2+13x-7}{x+7} \to \frac{0}{0}$

$\lim_{x\to-7} \frac{(2x-1)\cancel{(x+7)}}{\cancel{(x+7)}}$

$2(-7)-1 = \boxed{-15}$

$2x^2+13x-7$ $\quad a\cdot c: -14$, $b: 13$

$2x^2+14x-1x-7$

$2x(x+7)-1(x+7)$

$(2x-1)(x+7)$

Limit Does Not Exist

5. $\lim_{x\to-6} \frac{x^2+4x+3}{x+6} \to \frac{(-6)^2+4(-6)+3}{-6+6} \to \frac{15}{0}$ ← limit does not exist (vertical asymptote exists at $x=-6$)

$(a-b)(a+b)=a^2-b^2$ "a+b" is the conjugate of a−b, or the other half of the difference of squares.

Rationalize Fractions with Radicals

1. $\lim_{x\to5}\frac{\sqrt{x+4}-3}{x-5} \to \frac{\sqrt{9}-3}{5-5} \to \frac{0}{0}$ (indeterminate form)

$$\lim_{x\to5}\frac{\sqrt{x+4}-3}{x-5}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$$

$$\lim_{x\to5}\frac{x+4-3\sqrt{x+4}+3\sqrt{x+4}-9}{(x-5)(\sqrt{x+4}+3)}$$

$$\lim_{x\to5}\frac{\cancel{x-5}(1)}{\cancel{(x-5)}(\sqrt{x+4}+3)} \to \frac{1}{\sqrt{5+4}+3} \to \boxed{\frac{1}{6}}$$

2. $\lim_{x\to10}\frac{x-10}{3-\sqrt{x-1}} \to \frac{0}{0}$

$$\lim_{x\to10}\frac{x-10}{3-\sqrt{x-1}}\cdot\frac{3+\sqrt{x-1}}{3+\sqrt{x-1}}$$

$$\lim_{x\to10}\frac{(x-10)(3+\sqrt{x-1})}{9-(x-1)} \to \lim_{x\to10}\frac{(x-10)(3+\sqrt{x-1})}{10-x}$$

$$\lim_{x\to10}\frac{\cancel{(x-10)}(3+\sqrt{x-1})}{-1\cancel{(-10+x)}} \to \frac{3+\sqrt{9}}{-1} = \frac{6}{-1} = \boxed{-6}$$

Complex Fractions (create common denominator with fractions)

3. $\lim_{x\to0}\frac{x}{\frac{1}{x-4}+\frac{1}{4}} \to \frac{0}{0}$

$$\frac{4}{4(x-4)}+\frac{x-4}{4(x-4)}$$

$$\lim_{x\to0}\frac{x}{\frac{x}{4(x-4)}}$$

$$\lim_{x\to0}\frac{x}{1}\cdot\frac{4(x-4)}{x}$$

$$\lim_{x\to0}4(x-4) = 4(-4) = \boxed{-16}$$

4. $\lim_{x\to0}\frac{\frac{1}{(x+3)^2}-\frac{1}{9}}{x} \to \frac{0}{0}$

$$\frac{\frac{9\cdot1}{9(x+3)^2}-\frac{(x+3)^2\cdot1}{9(x+3)^2}}{x}$$

$$\lim_{x\to0}\frac{\frac{9-(x+3)^2}{9(x+3)^2}}{x}$$

$$\lim_{x\to0}\frac{9-(x^2+6x+9)}{9(x+3)^2}\cdot\frac{1}{x}$$

$$\lim_{x\to0}\frac{\cancel{9}-x^2-6x-\cancel{9}}{9(x+3)^2\cdot x} \to \frac{\cancel{x}(-x-6)}{9(x+3)^2\cdot\cancel{x}}$$

$$\frac{-0-6}{9(0+3)^2} \to \frac{-6}{81}$$

10. $\lim_{x\to1}\frac{\frac{1}{3}-\frac{1}{3x}}{x-1} \to \frac{0}{0}$

$$\lim_{x\to1}\frac{\frac{x\cdot1}{3x}-\frac{1}{3x}}{x-1}$$

$$\lim_{x\to1}\frac{\frac{x-1}{3x}}{x-1}$$

$$\lim_{x\to1}\frac{\cancel{x-1}}{3x}\cdot\frac{1}{\cancel{x-1}}$$

$$\lim_{x\to1}\frac{1}{3x} \to \frac{1}{3(1)} = \boxed{\frac{1}{3}}$$

11. $\lim_{x\to0}\frac{\sqrt{x+11}-\sqrt{11}}{x} \to \frac{0}{0}$

$$\lim_{x\to0}\frac{\sqrt{x+11}-\sqrt{11}}{x}\cdot\frac{(\sqrt{x+11}+\sqrt{11})}{(\sqrt{x+11}+\sqrt{11})}$$

$$\lim_{x\to0}\frac{x+11-11}{x(\sqrt{x+11}+\sqrt{11})} \to \lim_{x\to0}\frac{\cancel{x}(1)}{\cancel{x}(\sqrt{x+11}+\sqrt{11})} \to \frac{1}{\sqrt{11}+\sqrt{11}}$$

$$\boxed{\frac{1}{2\sqrt{11}}}$$

12. $\lim_{x\to3}\frac{\sqrt{2x-6}}{x} \to \frac{\sqrt{3(2)-6}}{3} \to \frac{0}{3} = \boxed{3}$

13. $\lim_{x\to0}\frac{\frac{1}{(x+2)^2}-\frac{1}{4}}{x}$

$$\frac{\frac{4\cdot1}{4(x+2)^2}-\frac{(x+2)^2\cdot1}{4(x+2)^2}}{x}$$

$$\lim_{x\to0}\frac{\frac{4-(x+2)^2}{4(x+2)^2}}{x}$$

$$\lim_{x\to0}\frac{\frac{4-(x^2+4x+4)}{4(x+2)^2}}{x}$$

$$\lim_{x\to0}\frac{\frac{\cancel{4}-x^2-4x-\cancel{4}}{4(x+2)^2}}{x} \to \frac{\frac{x(-x-4)}{4(x+2)^2}}{x}$$

$$\lim_{x\to0}\frac{\cancel{x}(-x-4)}{4(x+2)^2}\cdot\frac{1}{\cancel{x}} \to \frac{-4}{4(2)^2} = \boxed{\frac{-1}{4}}$$

14. $\lim_{x \to b} \frac{b-x}{\sqrt{x}-\sqrt{b}}$ is $\rightarrow \frac{b-b}{\sqrt{b}-\sqrt{b}} \rightarrow \frac{0}{0}$

$\boxed{\text{(A)}}$ $-2\sqrt{b}$ (B) $-\sqrt{b}$ (C) $2b$ (D) $\sqrt{b}$ (E) $2\sqrt{b}$

$$\lim_{x \to b} \frac{b-x}{\sqrt{x}-\sqrt{b}} \cdot \frac{(\sqrt{x}+\sqrt{b})}{(\sqrt{x}+\sqrt{b})} \rightarrow \lim_{x \to b} \frac{(b-x)(\sqrt{x}+\sqrt{b})}{x-b} \rightarrow \lim_{x \to b} \frac{\cancel{(b-x)}(\sqrt{x}+\sqrt{b})}{-1\cancel{(-x+b)}} \rightarrow \frac{\sqrt{b}+\sqrt{b}}{-1} = \boxed{-2\sqrt{b}}$$

Use the piecewise functions to find the given values.

15)

$$g(x) = \begin{cases} \sqrt{5-x}, & x < -4 \\ x^2-5, & -4 \le x < 2 \\ x-3, & x \ge 2 \end{cases}$$

a. $\lim_{x \to 2^-} g(x) =$

$2^2 - 5 = \boxed{-1}$

b. $\lim_{x \to -4^+} g(x) =$

$(-4)^2 - 5 = \boxed{11}$

c. $g(2) =$

$2 - 3 = \boxed{-1}$

d. $\lim_{x \to -4^-} g(x) =$

$\sqrt{5-(-4)} = \sqrt{9} = \boxed{3}$

e. $\lim_{x \to 2^+} g(x) =$

$(2) - 3 = \boxed{-1}$

f. $\lim_{x \to 2} g(x) =$

$\boxed{-1}$

g. $\lim_{x \to -4} g(x) =$

does not exist b/c

$\lim_{x \to -4^+} g(x) \neq \lim_{x \to -4^-} g(x)$

h. $g(-4) =$ $(-4)^2 - 5 = \boxed{11}$

16) <u>Limits of absolute value functions</u>

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} \rightarrow \frac{|1.9-2|}{1.9-2} \rightarrow \frac{0.1}{-0.1} \rightarrow \boxed{-1}$$

test $x = 1.9$