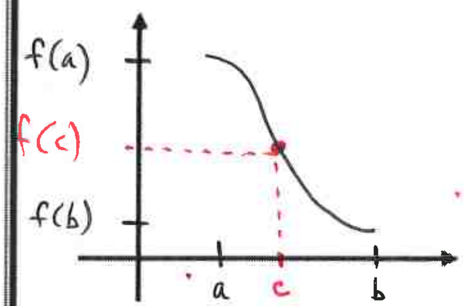
Key

**Intermediate Value Theorem (for continuous functions) - IVT**




Justification with the IVT.

1. The function $f(x)$ is continuous on an interval $[a, b]$.
2. $f(a) < f(b)$ or $f(b) > f(a)$.
3. $f(c)$ is between $f(a)$ and $f(b)$.

**Conclusion:** "According to the IVT, there is a value $c$ such that $f(c) =$ ___ and $a \le c \le b$."

Below is a table of values for a continuous function $f$.

| $x$ | 0 | 3 | 4 | 8 | 9 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | −5 | 3 | 7 | −1 |

1. On the interval $0 \le x \le 9$ what is the minimum number of zeros? $(y = 0)$

$\boxed{3}$

2. On the interval $4 \le x \le 9$, what is the fewest possible times $f(x) = 1$?

$\boxed{\text{one time}}$

3. On the interval $0 \le x \le 4$, ***must*** there be a value of $x$ for which $f(x) = 2$? Explain.

$f(0) = 1$
$f(4) = 3$

By IVT, since $f(x)$ is continuous on $[0, 4]$, there is a value $c$ such that $f(c) = 2$ since $f(0) = 1 < 2 < 3 = f(4)$

4. On the interval $4 \le x \le 8$, ***could*** there be a value of $x$ for which $f(x) = -2$? Explain.

By IVT, there is no guarantee that $f(x) = -2$. (since $f(4)$ and $f(8)$ are above $y = -2$)

5. Will the function $f(x) = x^2 - x + 1$ ever equal 8 on the interval $[-1, 5]$? Explain.

$f(x)$ continuous $[-1, 5]$
$f(-1) = 3$, $f(5) = 21$

By IVT, there is a value $c$ such that $f(c) = 8$ on interval $[-1, 5]$, since $f(-1) = 3 < 8 < 21 = f(5)$

**Use the Intermediate Value Theorem to answer each problem.**

16. If $f(x) = 3 - x^2$, will $f(x) = 0$ on the interval $[-2, 1]$? Explain.

$f(x)$ is continuous on $[-1, 1]$
$f(-2) = -1$
$f(1) = 2$

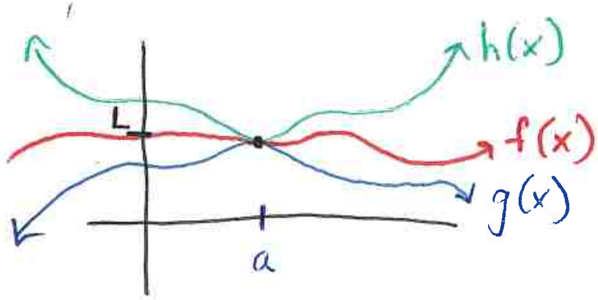
By IVT, there is a value $c$ such that $f(c) = 0$ on $[-1, 1]$ since $f(-2) = -1 < 0 < 2 = f(1)$

17. If $g(x) = \frac{1}{x}$, will $g(x) = -1$ on the interval $[2, 5]$? Explain. $g(x)$ is continuous on $[2, 5]$

$g(2) = \frac{1}{2}$
$g(5) = \frac{1}{5}$

IVT does not apply since both endpts have y-values greater than $g(x) = -1$

## 1.4 – Squeeze Theorem




**Squeeze Theorem:** a.k.a. "Sandwich Theorem" or "Pinching Theorem"

If $g(x) \le f(x) \le h(x)$

and if $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$

then $\lim_{x \to a} f(x) = L$

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**1. Find $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right)$**

$-1 \le \cos\left(\frac{1}{x^2}\right) \le 1$

$-x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2$

$\lim_{x \to 0} -x^2 \le \lim_{x \to 0} x^2\left(\cos\frac{1}{x^2}\right) \le \lim_{x \to 0} x^2$

$0 \le \lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le 0$

$\boxed{\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0}$

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**2. Let $g$ and $h$ be the functions defined by $g(x) = -x^2 + 2x - 3$ and $h(x) = 2x + 1$. If $f$ is a function that satisfies $g(x) \le f(x) \le h(x)$ for all $x$, what is $\lim_{x \to 2} f(x)$?**

$\lim_{x \to 2} g(x) \le \lim_{x \to 2} f(x) \le \lim_{x \to 2} h(x)$

$-3 \le \lim_{x \to 2} f(x) \le 5$

$\boxed{\lim_{x \to 2} f(x) \text{ is inconclusive by Squeeze theorem}}$

**3. Let $g$ and $h$ be the functions defined by $g(x) = \cos\left(\frac{\pi}{2}x\right) + 2$ and $h(x) = x^2 + 3$. If $f$ is a function that satisfies $g(x) \le f(x) \le h(x)$ for $-1 \le x \le 5$, what is $\lim_{x \to 0} f(x)$?**

$\lim_{x \to 0} \cos\left(\frac{\pi}{2}x\right) + 2 \le \lim_{x \to 0} f(x) \le \lim_{x \to 0} x^2 + 3$

$3 \le \lim_{x \to 0} f(x) \le 3$

$\boxed{\text{By squeeze theorem, } \lim_{x \to 0} f(x) = 3}$

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**4. Let $g$ and $h$ be the functions defined by $g(x) = x^2 - 3x$ and $h(x) = 2 - 2x$. If $f$ is a function that satisfies $g(x) \le f(x) \le h(x)$ for all $x$, what is $\lim_{x \to 2} f(x)$?**

$\lim_{x \to 2} x^2 - 3x \le \lim_{x \to 2} f(x) \le \lim_{x \to 2} 2 - 2x$

$-2 \le \lim_{x \to 2} f(x) \le -2$

$\boxed{\text{By Squeeze Theorem } \lim_{x \to 2} f(x) = -2}$

**5. Let $g$ and $h$ be the functions defined by $g(x) = \cos(\pi(x + 2)) - 3$ and $h(x) = \frac{x^2}{2} + x - \frac{7}{2}$. If $f$ is a function that satisfies $g(x) \le f(x) \le h(x)$ for $-2 \le x \le 0$, what is $\lim_{x \to -1} f(x)$?**

$\lim_{x \to -1} \cos[\pi(x+2)] - 3 \le \lim_{x \to -1} f(x) \le \lim_{x \to -1} \frac{x^2}{2} + x - \frac{7}{2}$

$-4 \le \lim_{x \to -1} f(x) \le \frac{-8}{2} = -4$

$\boxed{\text{By Squeeze Theorem, } \lim_{x \to -1} f(x) = -4}$