AP Calculus – 1.5 Notes Infinite Limits (Vertical Asymptotes) & Limits at Infinity (Horizontal Asymptotes)

Key

## I. Infinite Limits and Vertical Asymptotes

Use the function $f(x) = \frac{x^2+2x-8}{x^2+x-12}$ to answer the following.

**1. Identify all vertical asymptotes.**

$\frac{(\cancel{x+4})(x-2)}{(\cancel{x+4})(x-3)}$

hole at $x=-4$

$\boxed{\text{vertical asymptote at } x=3}$

**2. Evaluate $\lim_{x\to 3^-} f(x)$**

$\lim_{x\to 3^-} f(x) = \boxed{-\infty}$

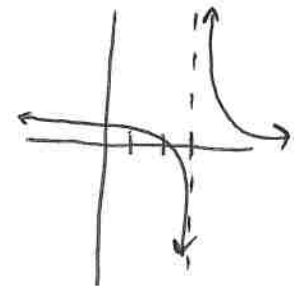
$\lim_{x\to 3^-} \frac{x-2}{x-3}$

test $x=2.9 \to \frac{+}{-} = \boxed{-\infty}$

**3. Evaluate $\lim_{x\to 3^+} f(x)$**

$\lim_{x\to 3^+} f(x) = \boxed{+\infty}$

$\lim_{x\to 3^+} \frac{x-2}{x-3} \to \frac{+}{+} = \boxed{+\infty}$



**Find the limit.**

**4.** $\lim_{x\to 3^+} \frac{1-x}{x-3} \to \frac{-2}{0}$ V.A. at $x=3$

test $x=3.1$ $\quad \frac{1-3.1}{3.1-3} \to \frac{-}{+} = \boxed{-\infty}$

**5.** $\lim_{x\to 1} \frac{x-3}{x^2-2x+1} \to \frac{-2}{0} \to$ VA at $x=1$

$\lim_{x\to 1^-} \frac{x-3}{(x-1)^2} \to$ test $x=0.9 \to \frac{-}{+} = \boxed{-\infty}$

$\lim_{x\to 1^+} \frac{x-3}{(x-1)^2} \to \frac{-}{+} = \boxed{-\infty}$

$\lim_{x\to 1} f(x) = \boxed{+\infty}$ (subset of does not exist)

**Practice Problems:**

**Identify the vertical asymptotes of each function.**

**1.** $f(x) = \frac{x-6}{x^2-9x+18}$

$f(x) = \frac{\cancel{x-6}}{(\cancel{x-6})(x-3)}$

$x-3=0$

$\boxed{VA: x=3}$

**2.** $f(x) = \frac{2x^2-x-3}{3x^2+4x+1}$

$f(x) = \frac{(2x-3)(\cancel{x+1})}{(3x+1)(\cancel{x+1})}$

$3x+1=0$

$x=-1/3$

$\boxed{VA: x=-1/3}$

**3.** $f(x) = \frac{x^2-x-12}{x+7} \to \frac{(x-4)(x+3)}{(x+7)}$

$\boxed{VA: x=-7}$

**4.** $f(x) = \frac{3x^2-11x+10}{x-2}$

$f(x) = \frac{(3x-5)(\cancel{x-2})}{(\cancel{x-2})}$

$\boxed{\text{No vertical asymptote}}$

**5.** $f(x) = \frac{x^3+2x^2-24x}{x^2-x}$

$f(x) = \frac{x(x^2+2x-24)}{x(x-1)}$

$f(x) = \frac{\cancel{x}(x+6)(x-4)}{\cancel{x}(x-1)}$

$\boxed{VA: x=1}$

**6.** $f(x) = \frac{7x^2+4x-3}{7x-3}$

$f(x) = \frac{(\cancel{7x-3})(x+1)}{\cancel{7x-3}}$

$\boxed{\text{No vertical asymptotes}}$

## II. Limits at Infinity and Horizontal Asymptotes

**Horizontal Asymptotes:** (End-behavior)

What does the $y$-value approach as the $x$-value approaches negative infinity AND positive infinity? Does it approach a specific number, or is it growing without bound?

**Checking for Horizontal Asymptotes (H.A.)** $\left(\lim_{x\to\infty} f(x) \quad or \quad \lim_{x\to-\infty} f(x)\right)$

If $f(x) = \frac{p(x)}{q(x)}$, then **compare the degrees between numerator and denominator**

i) If Numerator degree < Denominator degree, then the H.A. is y = 0

Example 1: $\lim_{x\to\infty} \frac{3x^2-7}{2x^3+1} = \boxed{0}$

ii) If Denominator degree = Numerator degree, then H.A. is $y = \frac{numerator\ coefficient}{denominator\ coefficient}$

Example 2: $\lim_{x\to\infty} \frac{5x^2+3}{2x^2+4x-9} = \boxed{\frac{5}{2}}$

iii) If Numerator degree > Denominator degree, then H.A. does not exist (limit is therefore $+\infty$ or $-\infty$)

Example 3: $\lim_{x\to\infty} \frac{2x^3+1}{7x^2+5x+10} = +\infty$ or $-\infty$ $\quad \frac{+}{+} = \boxed{+\infty}$

test $x=100$

Note: a H.A. is a description of end behavior, not a boundary that the graph can't cross. A function can NEVER cross a vertical asymptote, but it might cross a horizontal asymptote.

*same degrees, take ratio of coefficients

**Use Horizontal Asymptote Rules for the following:**

N>D

4) $\lim_{x\to\infty} \frac{3x^2+1}{2x-5}$ $\to +\infty$ / $-\infty$

*test $x=100$

$\frac{3(100)^2+1}{2(100)-5} \to \frac{+}{+} \to \boxed{+\infty}$

N>D

5) $\lim_{x\to-\infty} \frac{3x^2+1}{2x-5}$ $\to +\infty$ / $-\infty$

*test $x=-100$

$\frac{3(-100)^2+1}{2(-100)-5} \to \frac{+}{-} \to \boxed{-\infty}$

N=D

6) $\lim_{x\to-\infty} \frac{3x+1}{5-2x} \to \frac{3}{-2}$

$\to \boxed{-\frac{3}{2}}$

N=D

7) $\lim_{x\to\infty} \frac{3x+1}{5-2x} = \boxed{-\frac{3}{2}}$

N<D

8) $\lim_{x\to\infty} \frac{3x+1}{2x^2-5} = \boxed{0}$

N>D

9) $\lim_{x\to-\infty} \frac{3x^3+1}{2x^2-5}$

*test $x=-100$

$\frac{3(-100)^3+1}{2(-100)^2-5} \to \frac{-}{+} \to \boxed{-\infty}$

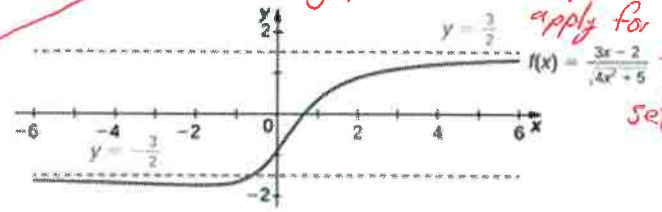
## B. Finding Horizontal Asymptotes with Radicals in denominator

Think of this as a special case. Horizontal Asymptotes that are split only apply for this type of setup.

Ex. 10: Find the Horizontal asymptotes for:

$$y = \frac{3x-2}{\sqrt{4x^2+5}}$$

*compare degrees




*Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} \to \frac{3}{\sqrt{4}} = \frac{3}{2}$$

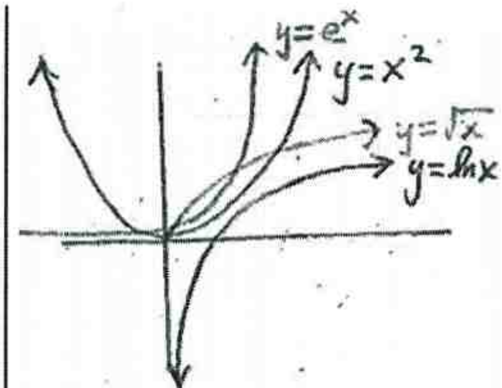
$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{4x^2+5}} \to \frac{-3}{\sqrt{4}} \to \frac{-3}{2}$$

$\boxed{\text{Horizontal Asymptotes are } y = \frac{3}{2} \text{ and } y = -3/2}$

## C. Comparative Growth Rates

*Families of Functions grow at predictable rates in relations to each other as x **approaches** $+\infty$

***Logarithms < Radicals < Polynomial (Algebraic) < Exponential**
(slowest) (fastest)




$$\lim_{x\to\infty} \frac{\text{slower}}{\text{faster}} = \boxed{0}$$

$$\lim_{x\to\infty} \frac{\text{faster}}{\text{slower}} = \boxed{+\infty \text{ or } -\infty}$$

test using $x=100$ to determine between $+\infty$ and $-\infty$

*Note: Comparative Growth Rates relationship <u>only apply</u> when limit approaches infinity. (NOT $-\infty$)

**Ex. 11** $\lim_{x\to\infty} \frac{\sqrt{5000x+1000}}{x^2} \to \frac{\text{Radical}}{\text{polynomial}}$

$= \boxed{0}$

**Ex. 13** $\lim_{x\to\infty} \frac{\ln(40000000x)}{2x} \to \frac{\text{logarithm}}{\text{algebraic}} = \boxed{0}$

**Ex. 12** $\lim_{x\to\infty} \frac{-e^{2x}}{1000x^4+x^5} \to \frac{\text{exponential}}{\text{polynomial}}$

test $x=100 \to \frac{-}{+}$

$\to \boxed{-\infty}$

**Ex. 14** $\lim_{x\to\infty} \frac{-\sqrt{3000x-4}}{\ln(5x+1)} \to \frac{\text{Radical}}{\text{Logs}}$

test $x=100$

$\to \frac{-}{+} \to \boxed{-\infty}$

## Unit Review Problems

5. If the function $f$ is continuous for all real numbers and if $f(x) = \frac{x^2+6x+8}{x+2}$ when $x \neq -2$, then $f(-2) =$

$$\lim_{x\to -2} \frac{x^2+6x+8}{x+2} \to \frac{0}{0} \to \lim_{x\to -2} \frac{(x+4)\cancel{(x+2)}}{\cancel{(x+2)}} = 2$$

$\boxed{f(-2)=2}$

6. Let $f$ be the function defined by $f(x) = \begin{cases} \frac{x^2+8x+12}{x+6}, & x \neq -6 \\ b, & x = -6 \end{cases}$. For what value of $b$ is $f$ continuous at $x = -6$?

$$\lim_{x\to -6} \frac{x^2+8x+12}{x+6} \to \frac{0}{0} \to \lim_{x\to -6} \frac{\cancel{(x+6)}(x+2)}{\cancel{(x+6)}} = -4$$

$\boxed{b=-4}$

i) $f(-6)=b$

ii) $\lim_{x\to -6} f(x) = -4$

iii) $f(-6) = \lim_{x\to -6} f(x)$

$\boxed{b=-4}$

**Evaluate the limit.**

7. $\lim_{x\to\infty} \sin\left(\frac{x+3\pi x^2}{2x^2}\right)$

$\lim_{x\to\infty} \sin\left(\frac{3\pi}{2}\right) = \boxed{-1}$

8. $\lim_{x\to -5^-} \frac{-3}{25-x^2} \to \frac{-3}{0}$ VA at $x=-5$

$\lim_{x\to -5^-} \frac{-3}{25-x^2} \to \frac{-3}{-} = \boxed{+\infty}$

test $x=-5.1$

9. $\lim_{x\to\infty} \frac{\sin x}{x} = \boxed{0}$

10. $\lim_{x\to\infty} \frac{4x^5-2x^2+3}{3x^2+2x^5-x^4}$ *compare degrees

$\frac{4}{2} = \boxed{2}$

11. $\lim_{x\to -1} \frac{x^2+1}{x+1} \to \frac{2}{0}$ VA at $x=-1$

$\lim_{x\to -1^-} \frac{x^2+1}{x+1} \to \frac{+}{-} = -\infty$

$\lim_{x\to -1^+} \frac{x^2+1}{x+1} \to \frac{+}{+} = +\infty$

$\boxed{\text{does not exist}}$

12. $\lim_{x\to\infty} x^5 3^{-x}$

$\lim_{x\to\infty} \frac{x^5}{3^x} = \boxed{0}$

Comparative Growth Rate Relationship
L < R < P < E

13. Identify all horizontal asymptotes of $f(x) = \frac{\sqrt{16\ ^6+x^3+5x}}{5x^3-8x}$.

$y = \frac{\sqrt{16}}{5}$ and $y = \frac{-\sqrt{16}}{5}$

$\boxed{\text{H.A.} \quad y = \frac{4}{5} \text{ and } y = -4/5}$