10.3a Parametric Equations and Derivatives

p. 711-714 #1-86

$$* \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Find dy/dx

2) $x = \sqrt[3]{t}$ $\quad \frac{dx}{dt} = \frac{1}{3}t^{-2/3} = \frac{1}{3t^{2/3}}$

$y = 4 - t$ $\quad dy/dt = -1$

$$\frac{dy/dt}{dx/dt} = \frac{-1}{\frac{1}{3t^{2/3}}} = \boxed{-3t^{2/3}}$$

4) $x = 2e^{\theta}$ $\quad dx/dt = 2e^{\theta} \cdot 1$

$y = e^{-\theta/2}$ $\quad dy/dt = e^{-\theta/2} \cdot -\frac{1}{2}$

$$\frac{dy/dt}{dx/dt} = \frac{-\frac{1}{2}e^{-\theta/2}}{2e^{\theta}} = \frac{-1}{4e^{\theta}e^{\theta/2}} = \boxed{\frac{-1}{4e^{3\theta/2}}}$$

Find slope and concavity

6) $x = \sqrt{t}$ $\quad y = 3t - 1$ $\quad \underline{\underline{t = 1}}$

$\frac{dx}{dt} = \frac{1}{2}t^{-1/2}$ $\quad dy/dt = 3$

$$\frac{dy}{dx} = \frac{3}{\frac{1}{2}t^{-1/2}} = 6t^{1/2} \qquad \left.\frac{dy}{dx}\right|_{t=1} = 6(1)^{1/2} = \boxed{6}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(6t^{1/2})}{\frac{1}{2\sqrt{t}}} = \frac{6 \cdot \frac{1}{2}t^{-1/2}}{\frac{1}{2}t^{-1/2}} = \boxed{6}$$

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8) $x = t^2 + 5t + 4$ $\quad y = 4t$ $\quad t = 0$

$\frac{dx}{dt} = 2t + 5$ $\quad \frac{dy}{dt} = 4$

$$\frac{dy}{dx} = \frac{4}{2t+5} \qquad \left.\frac{dy}{dx}\right|_{t=0} = \frac{4}{2(0)+5} = \boxed{\frac{4}{5}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{4}{2t+5}\right]}{2t+5} \quad \frac{\frac{d}{dt}\left[4(2t+5)^{-1}\right]}{2t+5}$$

$$\frac{d^2y}{dx^2} = \frac{4(-1)(2t+5)^{-2}(2)}{2t+5} = \frac{-8}{(2t+5)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \frac{-8}{(2(0)+5)^3} = \boxed{\frac{-8}{125}} \text{ (concave down)}$$

10) $x = \cos\theta \quad y = 3\sin\theta \quad \theta = 0$

$\frac{dx}{d\theta} = -\sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta$

$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$

$\left.\frac{dy}{dx}\right|_{\theta=0} -3\cot 0 = \text{undefined}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[-3\cot\theta]}{-\sin\theta} = \frac{-3\csc^2\theta}{-\sin\theta}$

$\frac{d^2y}{dx^2} = \frac{3}{\sin^3\theta}$

$\left.\frac{d^2y}{dx^2}\right|_{\theta=0} = \frac{3}{(\sin 0)^3} = \text{undefined}$

12) $x = \sqrt{t} \quad y = \sqrt{t-1} \quad t = 2$

$\frac{dx}{dt} = \frac{1}{2}t^{-1/2}$

$\frac{dy}{dt} = \frac{1}{2}(t-1)^{-1/2}$

$\frac{dy}{dx} = \frac{\frac{1}{2}(t-1)^{-1/2}}{\frac{1}{2}(t)^{-1/2}} = \sqrt{\frac{t}{t-1}}$

$\left.\frac{dy}{dx}\right|_{t=2} = \sqrt{\frac{2}{2-1}} = \sqrt{2}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\sqrt{\frac{t}{t-1}}\right]}{\frac{1}{2}t^{-1/2}} = \frac{\cancel{\frac{1}{2}}\left(\frac{t}{t-1}\right)^{-1/2}\left[\frac{1(t-1)-t(1)}{(t-1)^2}\right]}{\cancel{\frac{1}{2}}t^{-1/2}}$

$\cancel{t}-1-\cancel{t}$

$\frac{d^2y}{dx^2} = \frac{-1}{(t-1)^{3/2}}$

$\left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{-1}{(2-1)^{3/2}} = \boxed{-1}$

concave down

14) $x = \theta - \sin\theta \quad y = 1-\cos\theta \quad \theta = \pi$

$\frac{dx}{d\theta} = 1-\cos\theta$

$\frac{dy}{d\theta} = \sin\theta$

$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} =$

$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = 0$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{\sin\theta}{1-\cos\theta}\right]}{1-\cos\theta} = \frac{\frac{\cos\theta(1-\cos\theta)-\sin\theta(+\sin\theta)}{(1-\cos\theta)^2}}{1-\cos\theta}$

$\cos\theta - \cos^2\theta - \sin^2\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{\cos\theta-1}{(1-\cos\theta)^2}}{1-\cos\theta} = \frac{d^2y}{dx^2} = \frac{-1}{(1-\cos\theta)^2}$

$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{-1}{(1-\cos\pi)^2} = \frac{-1}{(2)^2} = \boxed{-\frac{1}{4}}$

concave down

16) Find equation of tangent line

$x = 2-3\cos\theta \quad y = 3+2\sin\theta$

$\frac{dx}{dt} = 3\sin\theta \quad \frac{dy}{dt} = 2\cos\theta$

$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$

a) $(-1, 3)$

$-1 = 2-3\cos\theta$
$1 = \cos\theta$
$\theta = 0$

$\left.\frac{dy}{dx}\right|_{(-1,3)} =$

$\left.\frac{dy}{dx}\right|_{\theta=0} \frac{2}{3}\cot 0 = \text{undefined}$

$\boxed{\text{line: } x=-1}$

b) $(2, 5)$

$2 = 2-3\cos\theta$
$0 = \cos\theta$
$\theta = \pi/2$

$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{2}{3}\cot(\pi/2) = 0$

$\boxed{y=5}$

c) $\left(\frac{4+3\sqrt{3}}{2}, 2\right)$ $\quad \sin\theta = -1/2 \quad \theta = 7\pi/6$

$\left.\frac{dy}{dx}\right|_{\theta=7\pi/6} = \frac{2}{3}\frac{\cos 7\pi/6}{\sin 7\pi/6} = \frac{2}{3}\cdot\frac{\sqrt{3}/2}{1/2}$

$\left.\frac{dy}{dx}\right|_{\theta=7\pi/6} = \frac{2\sqrt{3}}{3}$

$\boxed{y-2 = \frac{2\sqrt{3}}{3}\left[x - \frac{4+3\sqrt{3}}{2}\right]}$

18) Write equation of tangent lines $x = t^4 + 2$, $y = t^3 + t$

$\frac{dx}{dt} = 4t^3 \qquad \frac{dy}{dt} = 3t^2 + 1$

$\frac{dy}{dx} = \frac{3t^2+1}{4t^3} = \frac{3t^2+1}{4t^3}$

a) (3,-2)

$3 = t^4 + 2 \quad t^4 = 1 \quad t = -1$

$-2 = t^3 + t \quad t = -1$

$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2+1}{-4} = \frac{4}{-4} = -1$

$\boxed{y + 2 = -1(x - 3)}$

b) (2,0)

$2 = t^4 + 2 \quad t = 0$

$\left.\frac{dy}{dx}\right|_{t=0} = \frac{3(0)+1}{0} = \text{undefined}$ line $\boxed{x = 2}$

c) (18,10)

$18 = t^4 + 2 \quad t^4 = 16 \quad t = 2$

$10 = t^3 + t \quad \underline{t = 2}$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3(2)^2+1}{4(2)^3} = \frac{13}{32}$

$\boxed{y - 10 = \frac{13}{32}(x - 18)}$

20) Find equation of tangent line

$x = t - 2 \qquad y = \frac{1}{t} + 3 \qquad t = 1$

$y = t^{-1} + 3$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = -1t^{-2}$

$\frac{dy}{dx} = \frac{-1t^{-2}}{1} = \frac{-1}{t^2}$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{-1}{1} = -1$

$x(1) = 1 - 2 = -1$

$y(1) = \frac{1}{1} + 3 = 4$

point $(-1, 4)$ slope: $m = -1$

$y - y_1 = m(x - x_1)$

$\boxed{y - 4 = -1(x + 1)}$

22) $x = 3t - t^2 \qquad y = 2t^{3/2} \qquad t = 1/4$

$\frac{dx}{dt} = 3 - 2t \qquad dy/dt = 2 \cdot \frac{3}{2} t^{1/2}$

$\frac{dy}{dx} = \frac{3t^{1/2}}{3 - 2t}$

$\left.\frac{dy}{dx}\right|_{t=1/4} = \frac{3(1/4)^{1/2}}{3 - 2(1/4)} = \frac{\frac{3}{2}}{5/2}$

$\left.\frac{dy}{dx}\right|_{t=1/4} = \frac{3}{5}$

$x(1/4) = 3(1/4) - (1/4)^2$
$= \frac{3}{4} - \frac{1}{16} = \frac{12}{16} - \frac{1}{16} = 11/16$

$y(1/4) = 2(1/4)^{3/2} = \frac{2}{8} = 1/4$

point: $(11/16, 1/4)$ $m = 3/5$

$\boxed{y - 1/4 = \frac{3}{5}(x - 11/16)}$

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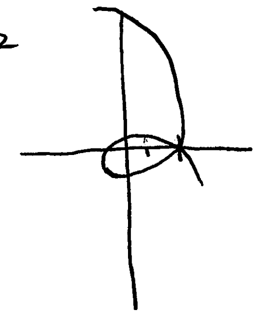
24) Find equation of tangent line: (where curve crosses itself)

$x = 2 - \pi\cos t \qquad y = 2t - \pi\sin t$

$2t - \pi\sin t = 2 - \pi\cos t$

$\boxed{\text{*crosses itself at } (2,0)}$ $\quad t = \pm\pi/2$



$\dfrac{dx}{dt} = +\pi\sin t \qquad \dfrac{dy}{dt} = 2 - \pi\cos t$

$\dfrac{dy}{dx} = \dfrac{2-\pi\cos t}{\pi\sin t}$ $\quad \left.\dfrac{dy}{dx}\right|_{t=\pi/2} = \dfrac{2-0}{\pi(1)} = \dfrac{2}{\pi}$

point: $(2,0)$

slope: $2/\pi$

$\boxed{y - 0 = \frac{2}{\pi}(x-2)}$

at $t = -\pi/2$

$\left.\dfrac{dy}{dx}\right|_{t=-\pi/2} = \dfrac{2-0}{\pi(-1)} = \dfrac{-2}{\pi}$

point: $(2,0)$ slope: $m = \frac{-2}{\pi}$

$\boxed{y - 0 = \frac{-2}{\pi}(x-2)}$

26) $x = t^3 - 6t \qquad y = t^2$ $\quad$ crosses itself: $(0,6)$

$0 = t(t^2-6) \qquad 6 = t^2$

$t = \pm\sqrt{6} \qquad t = \pm\sqrt{6}$

$\dfrac{dx}{dt} = 3t^2 - 6$ $\quad \boxed{\dfrac{dy}{dx} = \dfrac{2t}{3t^2-6}}$

$\dfrac{dy}{dt} = 2t$

at $t = \sqrt{6}$

$\left.\dfrac{dy}{dx}\right|_{t=\sqrt{6}} = \dfrac{2\sqrt{6}}{3(6)-6} = \dfrac{2\sqrt{6}}{12} = \dfrac{\sqrt{6}}{6}$

point: $(0,6)$

slope: $m = \sqrt{6}/6$

$\boxed{y - 6 = \frac{\sqrt{6}}{6}(x-0)}$

at $t = -\sqrt{6}$

$\left.\dfrac{dy}{dx}\right|_{t=-\sqrt{6}} = \dfrac{2(-\sqrt{6})}{3(6)-6} = \dfrac{-2\sqrt{6}}{12} = \dfrac{-\sqrt{6}}{6}$

point: $(0,6)$ slope: $m = \frac{-\sqrt{6}}{6}$

$\boxed{y - 6 = \frac{-\sqrt{6}}{6}(x-0)}$

Horizontal and Vertical Tangency

* Horizontal tangent occurs where $\frac{dy}{dx} = \frac{0}{\text{nonzero}}$, where $\boxed{\frac{dy}{dt} = 0}$

* Vertical tangent occurs where $\frac{dy}{dx} = \frac{\text{nonzero}}{0}$, where $\boxed{\frac{dx}{dt} = 0}$

28) $x = 2\theta \quad y = 2(1-\cos\theta)$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 0 - 2(-\sin\theta) \qquad \frac{dy}{dt} = 2\sin\theta \qquad \theta = 0, \pi, 2\pi, \pm n\pi$

$0 = 2\sin\theta$

Horizontal tangent set $\frac{dy}{dt} = 0$ → points: $(0,0), (2\pi, 4), (4\pi, 0)$,

vertical tangent: $\frac{dx}{dt} = 0$, $\frac{dx}{dt} = 2 \neq 0$, no vertical tangents on graph.

30) $x = t+1 \quad y = t^2 + 3t$

$\frac{dy}{dt} = 2t+3$ horizontal tangent: set $\frac{dy}{dt} = 0 \quad 2t+3 = 0 \quad t = -3/2$

point: $x(-3/2) = -1/2 \quad y(-3/2) = 9/4 - 9/2 = -9/4$ | point: $(-1/2, -9/4)$

slope: $m = 0$

$\boxed{y = -9/4}$

$\frac{dx}{dt} = t+1$ vertical tangent

$\frac{dx}{dt} = 1$, $0 \neq 1$ $\boxed{\text{no vertical tangents}}$

$\cancel{x(-1) = 0}$

$y\cancel{(-1) = 1 - 3 = -2}$

32) $x = t^2 - t + 2$ $\qquad y = t^3 - 3t$

$\frac{dx}{dt} = 2t - 1$ $\qquad \frac{dy}{dt} = 3t^2 - 3$

horizontal tangent:

set $\frac{dy}{dt} = 0$, $3t^2 - 3 = 0$, $t^2 = 1$ $\quad t = \pm 1$

$x(1) = 2$
$y(1) = 1 - 3 = -2$

$\boxed{\text{point: } (2, -2) \quad y = -2}$

$x(-1) = 4$
$y(-1) = 2$

$\boxed{\text{point: } (4, 2) \quad y = 2}$

vertical tangent:

set $\frac{dx}{dt} = 0$ $\quad 2t - 1 = 0$ $\quad t = 1/2$

$x(1/2) = 7/4$

$y(1/2) = -11/8$

$\boxed{\text{point: } (7/4, -11/8) \quad x = 7/4}$

34) $x = \cos\theta$ $\qquad y = 2\sin 2\theta$ $\qquad \frac{dx}{d\theta} = -\sin\theta$ $\qquad \frac{dy}{d\theta} = 2\cos 2\theta \cdot 2 = 4\cos 2\theta$

horizontal tangent:

set $\frac{dy}{d\theta} = 0$ $\quad 4\cos 2\theta = 0$
$\cos 2\theta = 0$
$2\theta = \cos^{-1}(0)$

$2\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$
$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$

points: $\left(\frac{\sqrt{2}}{2}, 2\right), \left(-\frac{\sqrt{2}}{2}, -2\right), \left(-\frac{\sqrt{2}}{2}, 2\right), \left(\frac{\sqrt{2}}{2}, -2\right)$

vertical tangent:

$\frac{dx}{d\theta} = -\sin\theta$

$-\sin\theta = 0$
$\theta = 0, \pi$

points: $(1, 0), (-1, 0)$

36) $x = 4\cos^2\theta$ $\quad y = 2\sin\theta$

$\frac{dx}{d\theta} = 8\cos\theta(-\sin\theta)$ $\quad \frac{dy}{d\theta} = 2\cos\theta$

horizontal tangent: $\frac{dy}{d\theta} = 0 = 2\cos\theta$ $\quad \theta = \cancel{\pi/2}, \cancel{3\pi/2}$, none

vertical tangent: $\frac{dx}{d\theta} = 0$, $-8\cos\theta\sin\theta = 0$ $\quad \theta = \cancel{\pi/2}, \cancel{3\pi/2}, \boxed{0, \pi}$

$\boxed{\text{point: } (4,0)}$

38) $x = \cos^2\theta$ $\quad y = \cos\theta$ $\quad \frac{dx}{d\theta} = 2\cos\theta(-\sin\theta)$ $\quad \frac{dy}{d\theta} = -\sin\theta$

$= -\sin 2\theta$

Horizontal tangent:

$\frac{dy}{d\theta} = 0$ $\quad -\sin\theta = 0$ $\quad \theta = \cancel{0}, \cancel{\pi}$ $\boxed{\text{none}}$

vertical tangent:

$\frac{dx}{d\theta} = 0$ $\quad -\sin 2\theta = 0$ $\quad 2\theta = \sin^{-1}(0)$

$2\theta = 0, \pi, 2\pi, 3\pi$

$\theta = \cancel{0}, \frac{\pi}{2}, \cancel{\pi}, \frac{3\pi}{2}, 2\pi$

$\boxed{\text{point: } (0,0)}$

40) Determine intervals concave up/down

$x = 2 + t^2$ $\quad y = t^2 + t^3$

$\frac{dx}{dt} = 2t$ $\quad \frac{dy}{dt} = 2t + 3t^2$

$\frac{dy}{dx} = \frac{2t+3t^2}{2t} = \frac{2+3t}{2} = 1 + \frac{3}{2}t$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[1+\frac{3}{2}t\right]}{2t} = \frac{3/2}{2t} = 3/4t$

concave up for $t > 0$

Concave down for $t < 0$

42) $x = t^2 \quad y = \ln t \qquad \frac{dx}{dt} = 2t \quad \frac{dy}{dt} = \frac{1}{t}$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{1}{2}t^{-2}\right]}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{2}\cdot -2t^{-3}}{2t} = \frac{-1}{2}t^{-4} = \boxed{\frac{-1}{2t^4}}$$

Domain: $t > 0$

graph is always concave down for $t > 0$

44) $x = 4\cos t \quad y = 2\sin t \quad 0 < t < 2\pi$

$$\frac{dx}{dt} = -4\sin t \qquad \frac{dy}{dt} = 2\cos t$$

$$\frac{dy}{dx} = \frac{2\cos t}{-4\sin t} = \frac{-1}{2}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{-1}{2}\cot(t)\right]}{-4\sin t} = \frac{+\frac{1}{2}\csc^2 t}{-4\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{8\sin^3 t}$$

$\cap \quad \cup$

$- \quad +$

$0 \quad \pi \quad 2\pi$

concave down $(0, \pi)$

concave up $(\pi, 2\pi)$