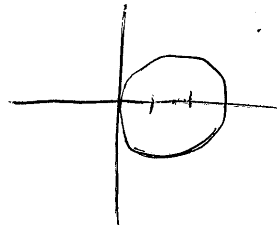
10.5 Polar Area p.731 #5-65 (Skp #48, #50)

Power-reducing identities
a) $\sin^2 u = \frac{1}{2}(1-\cos 2u)$
b) $\cos^2 u = \frac{1}{2}(1+\cos 2u)$

* $\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

Find Area of Polar region

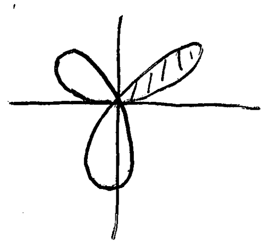
6) Interior of $r = 3\cos\theta$



$$A = \frac{1}{2}\int_0^\pi r^2\, d\theta = \frac{1}{2}\int_0^\pi (3\cos\theta)^2\, d\theta = \frac{9}{2}\int_0^\pi \cos^2 d\theta = \frac{9}{2}\int_0^2 \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = \boxed{\frac{9}{4}\pi}$$

8) One petal of $r = 4\sin 3\theta$

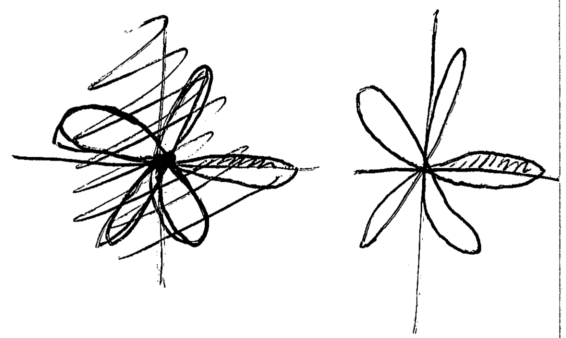


$[0, \pi]$

$$A = \frac{1}{2}\int_0^{\pi/3} [4\sin\theta]^2\, d\theta = \frac{1}{2}\int 16\sin^2\theta\, d\theta$$

$$8 \cdot \int \frac{1-\cos 6\theta}{2}\, d\theta = 4\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = 4(\pi/3) = \boxed{\frac{4\pi}{3}}$$

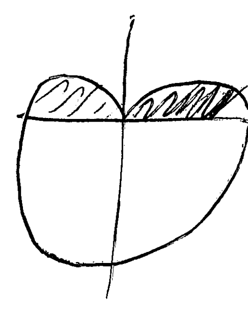
10) One petal of $r = \cos 5\theta$



$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/10} [\cos 5\theta]^2\, d\theta = \int \frac{1}{2}(1+\cos 10\theta)$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 10\theta}{10}\right]_0^{\pi/10} = \boxed{\frac{\pi}{20}}$$

12) Interior of $r = 1 - \sin\theta$ (above polar axis)



$$2 \cdot \frac{1}{2}\int_0^{\pi/2} [1-\sin\theta]^2\, d\theta$$

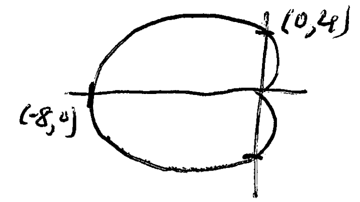
$$\int_0^{\pi/2} 1 - 2\sin\theta + \sin^2\theta\, d\theta$$

$$\int 1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\int \frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta$$

$$\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\Big]_0^{\pi/2} = \boxed{\frac{3\pi}{4} - 2}$$

14) Interior of $r = 4 - 4\cos\theta$



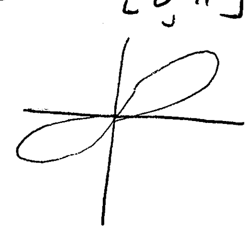
$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta \quad \text{or} \quad 2 \cdot \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$$

$$\frac{1}{2}\int_0^{2\pi} [4 - 4\cos\theta]^2 \, d\theta = \frac{1}{2}\int 16 - 32\cos\theta + 16\cos^2\theta \, d\theta \quad \frac{1}{2} \cdot 16 \int 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$8\int 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta = 8 \cdot \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 8\left(\tfrac{3}{2}\right)(2\pi) = \boxed{24\pi}$$

16) Interior of $r^2 = 6\sin 2\theta$ $\quad r = \pm\sqrt{6\sin 2\theta}$ $\quad [0, \pi]$

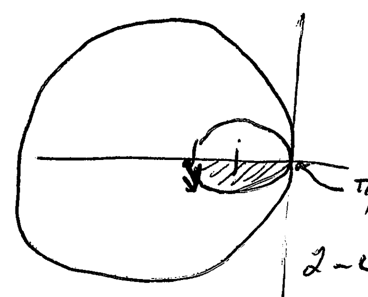
lemniscate



$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} \left(\sqrt{6\sin 2\theta}\right)^2 d\theta = \int_0^{\pi/2} 6\sin 2\theta \, d\theta$$

$$= 3 \cdot -\cos 2\theta \Big]_0^{\pi/2} = \boxed{6}$$

18) Inner loop of $r = 2 - 4\cos\theta$



$2 - 4\cos(\pi/3)$

$2 - 4(1/2)$

$2 - 2 = 0$ ✓

$$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} (2 - 4\cos\theta)^2 = \int 4 - 16\cos\theta + 16\cos^2\theta \, d\theta$$

$$4\int 1 - 4\cos\theta + 4\left[\tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$4\int 3 - 4\cos\theta + 2\cos 2\theta \, d\theta$$

$$4 \cdot 3\theta + 4\sin\theta + \sin 2\theta \Big]_0^{\pi/3} = \boxed{4\pi - 6\sqrt{3}}$$

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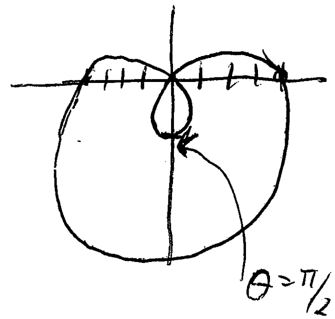
20) Inner loop of $r = 4 - 6\sin\theta$ $\quad \sin\theta = \frac{2}{3}$, $\theta = \sin^{-1}(2/3)$



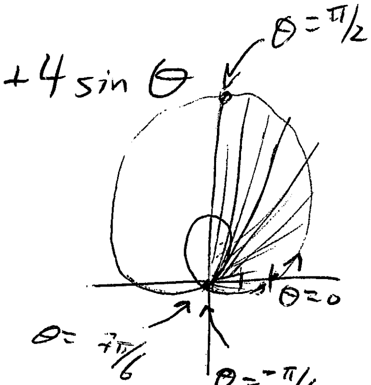
$$A = 2 \cdot \frac{1}{2}\int_{\arcsin 2/3}^{\pi/2} [4 - 6\sin\theta]^2 \, d\theta = \int 16 - 48\sin\theta + 36\sin^2\theta$$

$$\int 16 - 48\sin\theta + 36 \cdot \frac{1}{2}[1 - \cos 2\theta] \, d\theta$$

$$\int 16 - 48\sin\theta + 18 - 18\cos 2\theta = \int 34 - 48\sin\theta - 18\cos 2\theta \, d\theta$$

$$34\theta + 48\cos\theta - 9\sin 2\theta \Big]_{\arcsin 2/3}^{\pi/2} \approx \boxed{1.7635}$$

22) Between the loops of $r = 2(1 + 2\sin\theta) = 2 + 4\sin\theta$



$$\text{Outer loop Area} = 2 \cdot \frac{1}{2}\int_{-\pi/6}^{\pi/2} [2 + 4\sin\theta]^2 \, d\theta = 8\pi + 6\sqrt{3}$$

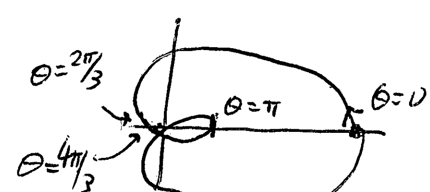
$$\text{inner loop Area} = 2 \cdot \frac{1}{2}\int_{7\pi/6}^{3\pi/2} [2 + 4\sin\theta]^2 \, d\theta = 4\pi - 6\sqrt{3}$$

$$\text{Outer Loop} - \text{Inner Loop Area} = 8\pi + 6\sqrt{3} - (4\pi - 6\sqrt{3})$$

$$\approx \boxed{4\pi + 12\sqrt{3}}$$

24) Between loops of $r = \frac{1}{2} + \cos\theta$

$$\text{Outer Loop} = 2 \cdot \frac{1}{2}\int_{0}^{2\pi/3} [1/2 + \cos\theta]^2 \, d\theta = \frac{\pi}{2} + \frac{3\sqrt{3}}{8}$$



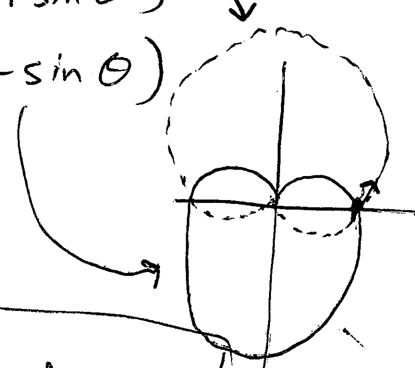
$$\text{Inner Loop} = 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi} [1/2 + \cos\theta]^2 \, d\theta = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$$

$$\text{Outer Loop} - \text{Inner Loop}: \frac{\pi}{2} + \frac{3\sqrt{3}}{8} - \left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \boxed{\frac{\pi}{4} + \frac{3\sqrt{3}}{4}}$$

26) Find points of Intersection

$r = 3(1+\sin\theta)$

$r = 3(1-\sin\theta)$



$3(1+\sin\theta) = 3(1-\sin\theta)$

$1+\sin\theta = 1-\sin\theta$

$2\sin\theta = 0 \qquad \theta = 0, \pi$

$\boxed{\text{points: } (3,0), (3,\pi), (0,0)}$

28) $r = 2-3\cos\theta$

$r = \cos\theta$

$2-3\cos\theta = \cos\theta$

$2 = 4\cos\theta \qquad \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$\boxed{\text{points: } \left(\frac{1}{2}, \frac{\pi}{3}\right), \left(\frac{1}{2}, \frac{5\pi}{3}\right)(0,0)}$

30) $r = 3+\sin\theta$

$r = 2\csc\theta$

$3+\sin\theta = 2\csc\theta$

$3+\sin\theta - 2\csc\theta = 0$

$3+\sin\theta - \frac{2}{\sin\theta} = 0$

$3\sin\theta + \sin^2\theta - 2 = 0$

$\sin^2\theta + 3\sin\theta - 2 = 0 \qquad x^2+3x-2=0$

$\cancel{(\sin\theta - 1)(\sin\theta - 2)}$ * quadratic formula

$\frac{-3 \pm \sqrt{9-4(1)(-2)}}{2} = \frac{-3 \pm \sqrt{17}}{2} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{\sqrt{17}-3}{2}\right) \boxed{\approx 0.596}$

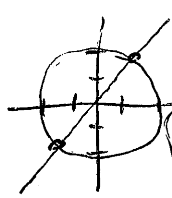
$(3.56, 0.596)$

$(3.56, 2.545)$

$\pi - 0.596$

32) $\theta = \pi/4$

$r = 2$



points:

$(2, \pi/4), (-2, \pi/4)$

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34) $r = 4\sin\theta$
$r = 2(1+\sin\theta)$

$4\sin\theta = 2 + 2\sin\theta$

$2\sin\theta = 2$

$\sin\theta = 1$

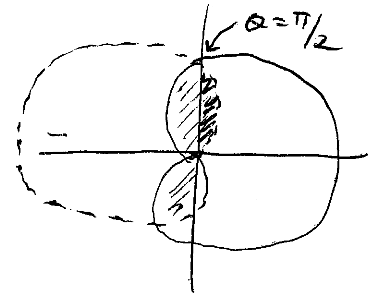
$\theta = \pi/2$

points

$\boxed{(0,0),\ (4, \pi/2)}$

36) Common interior of $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$

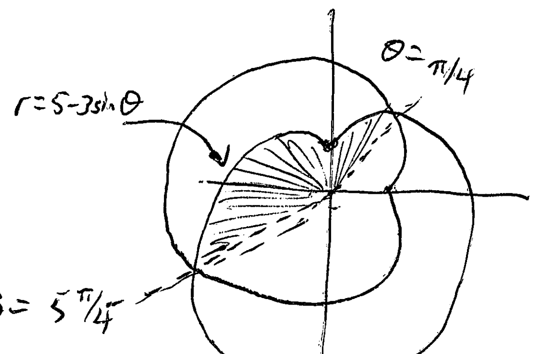
$$A = 4\cdot\frac{1}{2}\int_0^{\pi/2}\left[2-2\cos\theta\right]^2 d\theta = \boxed{6\pi - 16}$$



38) common interior of $r = 5 - 3\sin\theta$, $r = 5 - 3\cos\theta$

$$A = 2\cdot\frac{1}{2}\int_{\pi/4}^{5\pi/4}\left[5-3\sin\theta\right]^2 d\theta$$

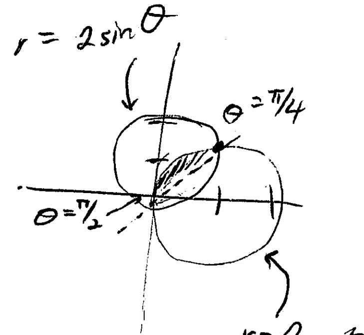
$$= \boxed{\frac{59\pi}{2} - 30\sqrt{2} \approx 50.251}$$



40) Common interior of $r = 2\cos\theta$ and $r = 2\sin\theta$

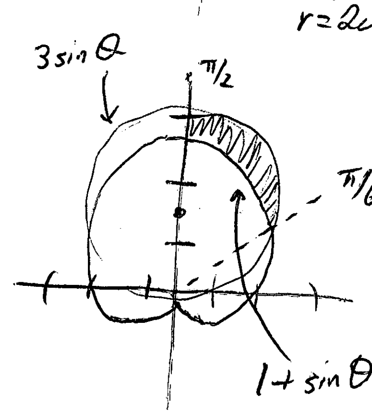
$$A = 2\cdot\frac{1}{2}\int_{\pi/4}^{\pi/2}\left[2\cos\theta\right]^2 d\theta \quad \text{or} \quad 2\cdot\frac{1}{2}\int_0^{\pi/4}\left[2\sin\theta\right]^2 d\theta$$

$$\boxed{A = \pi/2 - 1}$$

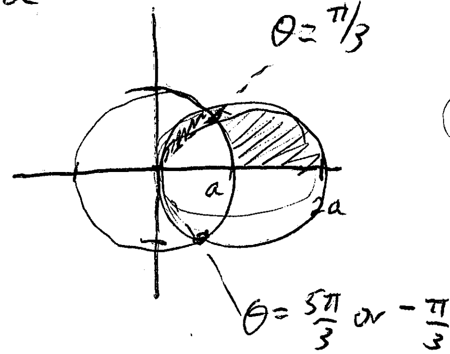


42) Inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left[3\sin\theta\right]^2 - \left[1+\sin\theta\right]^2 d\theta = \boxed{\pi}$$



44) Inside $r = 2a\cos\theta$ and outside $r = a$



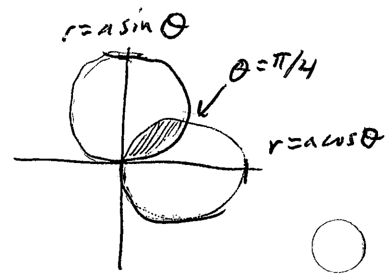
$2a\cos\theta = a$

$\cos\theta = 1/2$

$$A = \underbrace{\pi a^2}_{\text{area of } 2a\cos\theta} - \underbrace{\frac{\pi}{3}a^2}_{1/3 \text{ of circle } r=a} - \underbrace{2\cdot\frac{1}{2}\int_{\pi/3}^{\pi/2}\left[2a\cos\theta\right]^2 d\theta}_{\text{sliver of } 2a\cos\theta \text{ circle}}$$

$$= \boxed{\frac{2\pi a^2 + 3\sqrt{3}a^2}{6}}$$

46) Common interior of $r = a\cos\theta$ and $r = a\sin\theta$ where $a > 0$



$$A = 2\cdot\frac{1}{2}\int_0^{\pi/4}\left[a\sin\theta\right]^2 d\theta \quad \text{or} \quad 2\cdot\frac{1}{2}\int_{\pi/4}^{\pi/2}\left[a\cos\theta\right]^2 d\theta$$

$$= \boxed{\frac{1}{8}a^2\pi - \frac{1}{4}a^2}$$

Arc Length of Polar Curve

$$* \; s = \int_a^b \sqrt{[f(\theta)]^2 + f'(\theta)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

52) $r = a \qquad 0 \le \theta \le 2\pi$

$r' = 0$

$$s = \int_0^{2\pi}\sqrt{a^2 + 0^2}\, d\theta = \int a = a\theta\Big]_0^{2\pi} = \boxed{2\pi a}$$

54) $r = 2a\cos\theta \qquad -\pi/2 \le \theta \le \pi/2$

$r' = 2a(-\sin\theta) \qquad s = \int_{-\pi/2}^{\pi/2} \sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\, d\theta = \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta$

$s = 2a\int_{-\pi/2}^{\pi/2} \sqrt{1}\, d\theta = 2a\theta\Big]_{-\pi/2}^{\pi/2} = 2a(\pi/2) - -\left(2a(\pi/2)\right) = \boxed{2\pi a}$

56) $r = 8(1+\cos\theta) \qquad 0 \le \theta \le 2\pi$

$r' = 0 - 8\sin\theta = -8\sin\theta \qquad s = \int_0^{2\pi} \sqrt{(8+8\cos\theta)^2 + (-8\sin\theta)^2}\, d\theta$

$s = \boxed{64}$

58) $r = \sec\theta \qquad 0 \le \theta \le \pi/3 \qquad$ Length: $\boxed{1.73}$

60) $r = e^\theta \qquad 0 \le \theta \le \pi \qquad$ Length: $\boxed{31.31}$

62) $r = 2\sin(2\cos\theta) \qquad 0 \le \theta \le \pi \qquad$ Length: $\boxed{7.78}$

Area of Surface of Revolution

* About polar axis $\quad S = 2\pi \int_a^b f(\theta)\sin\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

* About $\theta = \pi/2$ $\quad S = 2\pi \int_a^b f(\theta)\cos\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

64) $r = a\cos\theta \qquad 0 \le \theta \le \pi/2 \qquad$ (About $\theta = \pi/2$)

$r' = -a\sin\theta$

$$S = 2\pi \int_0^{\pi/2} a\cos\theta \cdot \cos\theta \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta = \pi a^2 \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \pi a^2 \left(\theta + \frac{\sin 2\theta}{2}\right)\Big]_0^{\pi/2} = \boxed{\frac{\pi^2 a^2}{2}}$$

66) $r = a(1+\cos\theta) \qquad 0 \le \theta \le \pi \qquad$ (About polar axis)

$r' = -a\sin\theta$

$$S = 2\pi \int_0^{\pi} (a + a\cos\theta)(\sin\theta) \sqrt{(a + a\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$$

$$= -2\sqrt{2}\pi a^2 \int_0^{\pi} (1+\cos\theta)^{3/2}(-\sin\theta)\, d\theta = \frac{-4\sqrt{2}\pi a^2}{5}\Big[(1+\cos\theta)^{5/2}\Big]_0^{\pi} = \boxed{\frac{32\pi a^2}{5}}$$