Name: WS #1 KEY Date:__________ Period: _____

# End-of-Unit 10 Review – Infinite Sequences and Series

**Lessons 10.~~5~~6 through 10.10**

1. If the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^3 - 1}$ is approximated by $S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{2n^3 - 1}$, what is the least value of $k$ for which the alternating series error bound guarantees that $|S - S_k| < 10^{-3}$? $\rightarrow \frac{1}{10^3} \rightarrow \frac{1}{1000}$

$|S - S_k| \le |a_{k+1}| < \frac{1}{1000}$

$\frac{1}{2(k+1)^3 - 1} < \frac{1}{1000}$

$1000 < 2(k+1)^3 - 1$

$1001 < 2(k+1)^3$

$500.5 < (k+1)^3$

$7.939 < k+1$

$6.939 < k$

$\boxed{k > 6.939}$

(A) 6 $\boxed{\text{(B) } 7}$ (C) 8 (D) 9

2. If the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4n+1}$ is approximated by the partial sum with 15 terms, what is the alternating series error bound?

$\underbrace{\text{series error bound}}$ *next term $a_{n+1}$

$16^{th}$ term is $\frac{1}{4(16)+1} = \boxed{\frac{1}{65}}$

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{61}$ $\boxed{\text{(D) } \frac{1}{65}}$

3. Let $P(x) = 3 - 2x^2 + 5x^4$ be the fourth-degree Taylor Polynomial for the function $f$ about $x = 0$. What is the value of $f^{(4)}(0)$?

* $P_n(x) = \frac{f^n(c)}{n!}(x-c)^n$

$n = 4$, $c = 0$, $x = 0$

$f^4(x) = \frac{f^4(0)}{4!}(x-0)^4 = 5x^4$

$\frac{f^4(0)}{4!} = 5$

$f^4(0) = 5 \cdot 4! = \boxed{120}$

4. The function $f$ has derivatives of all orders for all real numbers with $f(2) = -2$, $f'(2) = 4$, $f''(2) = 8$, and $f'''(2) = 14$. Using the third-degree Taylor Polynomial for $f$ about $x = 2$, what is the approximation of $f(2.2)$?

$c = 2$

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$P_3(x) = -2 + 4(x-2) + \frac{8}{2!}(x-2)^2 + \frac{14}{3!}(x-2)^3$

$f(2.2) \approx -2 + 4(2.2-2) + 4(2.2-2)^2 + \frac{7}{3}(2.2-2)^3 = \boxed{-1.021}$

$C=0,\ x=1$ | *LaGrange $R_n(x) \le \dfrac{|\max(f^{n+1}(z))|}{(n+1)!}(x-c)^{n+1}$

5. Let $f$ be a function that has derivatives of all orders for all real numbers and let $P_4(x)$ be the fourth-degree Taylor Polynomial for $f$ about $x = 0$. $|f^{(n)}(x)| \le \frac{n}{n+1}$, for $1 \le n \le 6$ and all values of $x$. Of the following, which is the smallest value of $k$ for which the Lagrange error bound guarantees that $|f(1) - P_4(1)| \le k$?

$$R_4(x) \le \left|\frac{\max(f^5(z))}{5!}(1-0)^5\right| \le k$$

$$\le \frac{\left(\frac{5}{6}\right)}{5!}(1-0)^5 \rightarrow \frac{5}{6}\cdot\frac{1}{5!} \rightarrow \boxed{\frac{1}{6\cdot 4!} \le k}$$

(A) $\frac{4}{5}\left(\frac{1}{4!}\right)$ (B) $\frac{4}{5}\left(\frac{1}{5!}\right)$ $\boxed{\text{(C) } \frac{1}{6}\left(\frac{1}{4!}\right)}$ (D) $\frac{1}{6}\left(\frac{1}{5!}\right)$

6. The third Maclaurin polynomial for $\sin x$ is given by $f(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.3)$, what is the Lagrange error bound?

$* \dfrac{\max[f^{n+1}(z)]}{(n+1)!}(x-c)^{n+1}$

$* c=0$

$* x=0.3$

$* -1 \le \sin x \le 1$

$$\boxed{R_3(0.3) \le \frac{(1)}{4!}(0.3-0)^4 = 3.375\times10^{-4}}$$

7. Find the interval of convergence for the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$.

*Apply Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{(n+2)}\cdot\frac{n+1}{(x-1)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x-1)^n}\cdot(x-1)^2}{\cancel{(x-1)^n}\cdot(x-1)}\cdot\frac{n+1}{n+2}\right| < 1$$

$$\lim_{n\to\infty}|x-1| < 1$$

$-1 < x-1 < 1$

$0 < x < 2$

*Test endpoints!

test $x=0$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(0-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}\frac{(1)^{n+1}}{n+1} \text{ diverges}$$

test $x=2$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1} \text{ converges (AST)}$$

$$\boxed{0 < x \le 2}$$

8. If the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{5^n}$ is 5, what is the interval of convergence?

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-0)^n}{5^n}$$

$c=0$

$r=5$

$|x-c| < r$

$|x-0| < 5$

$|x| < 5$

$-5 < x < 5$

*test endpoints:

$x=-5$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-5)^n}{5^n} \rightarrow \frac{(5)^n}{5^n} \text{ diverges}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(5)^n}{5^n} \rightarrow (-1)^n \text{ diverges}$$

test $x=5$

$$\boxed{-5 < x < 5}$$

$$* \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

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$$* \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \frac{x^{2n-1}}{(2n-1)!}$$

9. Which of the following is an expression for a function $f$ that has the Maclaurin Series $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$?

series alternate signs

~~(A)~~ $\cos x$  ~~(B)~~ $e^x - \sin x$  $\boxed{\text{(C)} \; \frac{1}{2}(e^x + e^{-x})}$  ~~(D)~~ $e^{x^2} = 1 + (x^2) + \frac{x^4}{2!} + \dots$

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$$

$$1 + \frac{x^2}{2!} + \underbrace{2\left(\frac{x^3}{3!}\right)}$$

does not match series

$$\frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)\right]$$

$$\frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} \dots\right]$$

$$= \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} \checkmark$$

10. Find the Maclaurin Series for the function $f(x) = 2\sin x^3$. Write the first four non-zero terms.

$$* \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$2\sin(x^3) = 2\left[x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!}\right] = \boxed{2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \frac{2x^{21}}{7!}}$$

11. It is known the Maclaurin series for the function $\frac{1}{1+x}$ is defined by $\sum_{n=0}^{\infty}(-1)^n x^n$. Use this fact to find the first four nonzero terms and the general term for the power series expansion for $\frac{x^2}{1+x^2}$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots + (-1)^n (x)^n$$

$$\frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n (x)^{2n}$$

$$x^2 \cdot \frac{1}{1-(-x^2)} = x^2 - x^4 + x^6 - x^8 + \dots \quad (-1)^n x^{2n} \cdot x^2 \rightarrow \boxed{(-1)^n x^{2n+2}}$$

12. Let $T(x) = 7 - 3(x-3) + 5(x-3)^2 - 2(x-3)^3 + 6(x-3)^4$ be the fourth-degree Taylor Polynomial for the function $f$ about $x = 3$. Find the third-degree Taylor Polynomial for the derivative $f'$ about $x = 3$ and use it to approximate $f'(3.3)$.

$$T_3'(x) = -3(1) + 10(x-3) - 6(x-3)^2 + 24(x-3)^3$$

$$f'(3.3) \approx T_3'(3.3) = -3 + 10(3.3-3) - 6(0.3)^2 + 24(0.3)^3 = \boxed{0.108}$$

**Taylor Polynomial** is a polynomial that will approximate other function's values in a region that is nearby the "center"
*a tangent line is essentially a first degree taylor polynomial.

**$n^{th}$ degree taylor polynomial:**

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

**Alternating Series Remainder:**
Suppose an alternating series converges by AST. If the Series has Sum S, then $|R_n| = |S - S_n| \le |a_{n+1}|$

*This means that the **maximum error** for the $n^{th}$ term partial Sum $S_n$ is no greater than the absolute value of the first unused term $a_{n+1}$

**Taylor Series:** A General method for writing a power series representation for a function.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f^{(n)}$ represents the $n^{th}$ derivative evaluated at f.

**Maclaurin Series:** is the special case of Taylor series when $c = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

**Special Maclaurin Series:**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} \quad \text{IOC: All Reals}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} \quad \text{IOC: All Reals}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!} \quad \text{IOC: All Reals}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)} \quad \text{IOC}: -1 \le x \le 1$$

**Power Series:** Written in form $\sum_{n=0}^{\infty} a_n(x-c)^n$ where c and $a_n$ (coefficients) are numbers:
*Taylor and Maclaurin series are special cases of power series
For a power series centered at c, precisely one of the following is true:

1) The series converges only at c (ALL power series converge at least at their center) (Radius of convergence = 0)
2) The series converges for all x (function and infinite series have exact same values everywhere) → Radius = ∞
3) The series converges within a certain Radius of Convergence such that series converges for $|x - c| < R$
→The **interval of Convergence** (I.O.C.) is **[ ( c − R , c + R ) ]**

*Be sure to TEST convergence of endpoints
*Typically, you want to use the **RATIO TEST** to determine Radius of Convergence

**Geometric Series below based on**

$$S = \frac{a_1}{1-r} \quad \text{IOC: } -1 < x < 1$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots (-1)^n x^n + \ldots$$

IOC: $-1 < x < 1$

$$\frac{1}{x} = \frac{1}{1-[-(x-1)]} = 1 - (x-1) + (x-1)^2 - \ldots (-1)^n (x-1)^n$$

IOC: $0 < x < 2$

$$\ln x = \int \frac{1}{x}\,dx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \ldots \frac{(-1)^{n-1}(x-1)^n}{n}$$

IOC: $0 < x \le 2$

**LaGrange Error Bound** *This is similar to the Alternating Series Remainder. However, this method offers a way to determine the maximum error (remainder) when we do a Taylor polynomial approximation using a certain number of terms for a specific function.

$$R_n(x) = \left| \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right| \le \left| \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right|$$

* The remainder for an $n^{th}$ degree polynomial is found by taking the $(n+1)^{st}$ (**first unused**) derivative at "z" *We are not expected to find the exact value of z. (If we could, then an approximation would not be necessary) *We want to maximize the $(n+1)^{st}$ derivative on the interval from [x , c] in order to find a safe upper bound for the $\left|f^{(n+1)}(z)\right|$ *The maximum error bound is the worst case scenario for the interval in which our actual approximation can live. **College Board will provide strictly increasing and decreasing functions. (So we only have to choose between f( c) and f(x) (the endpoints). This will allow us to determine the max value much more accurately.

**Alternating Series Remainder:**
Suppose an alternating series converges by AST (such that $\lim_{n\to\infty} a_n = 0$ and $a_n$ is decreasing), then

$$|R_n| = |S - S_n| \le |a_{n+1}|$$

*This means that the **maximum error** for the $n^{th}$ term partial Sum $S_n$ is no greater than the absolute value of the first unused term $a_{n+1}$

Name: WS #2 KEY Date:

# End of Unit 10 WS#2 Infinite Sequences and Series

1. Let $f$ be the function defined by $f(x) = 3x\cos x$. What is the coefficient of $x^5$ in the Taylor Series for $f$ about $x = 0$? $(c=0)$ *product Rule

$\frac{f^n(c)}{n!}(x-c)^n$

$f'(x) = 3\cos x + -3x\sin x$

$f''(x) = -3\sin x - 3\sin x - 3x\cos x$

$= -6\sin x - 3x\cos x$

$f'''(x) = -6\cos x - 3\cos x + 3x\sin x$

$f'''(x) = -9\cos x + 3x\sin x$

$f^4(x) = 9\sin x + 3\sin x + 3x\cos x$

$= 12\sin x + 3x\cos x$

$f^5(x) = 12\cos x + 3\cos x - 3x\sin x$

$f^5(x) = 15\cos x - 3x\sin x$

$f^5(0) = 15\cos 0 - 3(0)\sin(0) = \boxed{15}$

5th degree term is $\frac{f^5(0)}{5!}(x-0)^5$

$\boxed{\frac{15}{5!}}x^5$

$\frac{3\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \boxed{\frac{1}{8}}$

2. Determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ with an error less than 0.0001.

$* \left|S - S_n\right| \le \left|a_{n+1}\right|$

*what makes $\left|\frac{1}{(n+1)^4}\right| < 0.0001$

$\frac{1}{(n+1)^4} < \frac{1}{10000}$

$(n+1)^4 > 10000$

$n+1 > \sqrt[4]{10000}$

$n+1 > 10$

$\boxed{n > 9}$

(A) 7 (B) 8 (C) 9 $\boxed{\text{(D) } 10}$

3. Find the third-degree Taylor Polynomial for the function $f(x) = \sqrt{x}$ about $x = 2$.

$* \frac{f^n(c)}{n!}(x-c)^n \rightarrow P_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\frac{\sqrt{2}}{16}}{2!}(x-2)^2 + \frac{\frac{3\sqrt{2}}{64}}{3!}(x-2)^3$

$\frac{\sqrt{2}}{16}\cdot\frac{1}{2} = \frac{\sqrt{2}}{32}$

$\frac{3\sqrt{2}}{64}\cdot\frac{1}{3\cdot 2} \rightarrow \frac{\sqrt{2}}{128}$

$f(x) = x^{1/2} \rightarrow f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}x^{-1/2} \rightarrow f'(x) = \frac{1}{2\sqrt{x}} \rightarrow f'(2) = \frac{1}{2\sqrt{2}}$ or $\frac{\sqrt{2}}{4}$

$f''(x) = \frac{-1}{4}x^{-3/2} \rightarrow f''(x) = \frac{-1}{4x^{3/2}} \rightarrow f''(2) = \frac{-1}{4(\sqrt{2})^3} = \frac{-1}{8\sqrt{2}} \rightarrow \frac{-\sqrt{2}}{16}$

$f'''(x) = \frac{3}{8}x^{-5/2} \rightarrow \frac{3}{8x^{5/2}} \rightarrow f'''(2) = \frac{3}{8(\sqrt{2})^5} \rightarrow \frac{3}{32\sqrt{2}} \rightarrow \frac{3\sqrt{2}}{64}$

$$\boxed{P_3(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3}$$

4. What is the coefficient of $x^3$ in the Maclaurin series for the function $\left(\frac{1}{1-x}\right)^2$?

$f(x) = \frac{1}{(1-x)^2} = (1-x)^{-2}$

$f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$

$f'''(x) = -24(1-x)^{-5}(-1) = 24(1-x)^{-5}$

$f'''(0) = 24(1-0)^{-5} = \frac{24}{(1)^5} = 24$

$\frac{f^3(0)}{3!}(x-0)^3 \rightarrow \boxed{\frac{24}{3!}}x^3$

$\frac{24}{3\cdot 2} = \frac{24}{6} = \boxed{4}$

5. The function $f$ has derivatives of all orders for all real numbers and $f^{(4)}(x) \leq \frac{1}{2}$. If a third-degree Taylor Polynomial for $f$ about $x = 0$ is used to approximate $f$ on $[0,1]$. What is the Lagrange error bound for the maximum error on interval $[0,1]$ in the approximation of $f(1)$?

$c = 0$, $x = 1$

*LaGrange Error Bound

$$R_n(x) \leq \frac{|\max(f^{n+1}(z))|}{(n+1)!}(x-c)^{n+1}$$

$$R_3(x) \leq \frac{|\max(f^4(x))|}{(3+1)!}(1-0)^4$$

$$R_3(1) \leq \frac{\left(\frac{1}{2}\right)}{4!}(1)^4 = \frac{1}{2}\cdot\frac{1}{4!} \rightarrow \frac{1}{2}\cdot\frac{1}{24} = \boxed{\frac{1}{48}}$$

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{24}$ $\boxed{\text{(D) } \frac{1}{48}}$

6. What is the alternating series error bound, if the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n+2}$ is approximated by the partial sum with 15 terms?

$$|S - S_n| \leq |a_{n+1}| \rightarrow a_{n+1} \rightarrow a_{15+1} \rightarrow \left|\frac{1}{5(15+1)+2}\right| \rightarrow \boxed{\frac{1}{82}}$$

* use the $16^{th}$ term ($a_{16}$) to determine the error bound of partial sum ($S_{15}$)

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{77}$ $\boxed{\text{(D) } \frac{1}{82}}$

7. $\max\limits_{0\leq x\leq 2}|f^{(5)}(x)| = 3.6$ $\max\limits_{0\leq x\leq 2}|f^{(6)}(x)| = 8.1$ $\max\limits_{0\leq x\leq 2}|f^{(7)}(x)| = 11.3$

Let $P(x)$ be the fifth-degree Taylor Polynomial for a function $f$ about $x = 0$. Information about the maximum of the absolute value of selected derivatives of $f$ over the interval $0 \leq x \leq 2$ is given in the table above. What is the smallest value of $k$ for which the Lagrange error bound guarantees that $|f(0.2) - P(0.2)| \leq k$?

$c = 0$
$x = 0.2$

$$R_5(x) = \frac{|\max f^6(x)|}{6!}(x-c)^6$$

$$R_5(0.2) = \frac{(8.1)}{6!}(0.2-0)^6 = (0.01125)(0.2)^6 = \boxed{7.2 \times 10^{-7}}$$

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8. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$. (I.o.C.)

*Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x-3)^n}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x-3)^n}(x-3)\cdot\cancel{3^n}}{\cancel{(x-3)^n}\cdot\cancel{3^n}\cdot 3}\cdot\frac{n}{n+1}\right|<1$$

$$\left|\frac{x-3}{3}\right|<1$$

$|x-c|<r$

$$|x-3|<3$$

$$-3<x-3<3$$

$$0<x<6$$

*test endpoints!

at $x=0$

$$\frac{(-1)^{n+1}(-3)^n}{n\cdot 3^n} \rightarrow \frac{(-1)^n(-3)^n\cdot(-1)^1}{n\cdot 3^n} \qquad (-1)^n(-1)^1(-3)^n$$

$$\rightarrow \frac{\cancel{(3)^n}(-1)^1}{n\cancel{(3^n)}} \rightarrow \frac{-1}{n}$$

(diverges)

at $x=6$

$$\frac{(-1)^{n+1}\cancel{(3)^n}}{n\cdot\cancel{3^n}} \rightarrow \frac{(-1)^{n+1}}{n}$$ (converges)

IoC: $\boxed{0<x\leq 6}$

9. What is the $\boxed{\text{coefficient}}$ of $(x-2)^4$ in the Taylor Polynomial for $f(x)=e^{4x}$ about $x=2$ ? $c=2$

$f(x)=e^{4x}$

$f'(x)=e^{4x}(4)$

$f''(x)=4^2e^{4x}$

$f'''(x)=4^3e^{4x}$

$f^4(x)=4^4e^{4x}$

$$*P_n(x)=\frac{f^n(c)}{n!}(x-c)^n$$

$$f^4(2)=4^4e^{4(2)}=4^4e^8$$

$$\frac{f^4(2)}{4!}(x-2)^4$$

$$\boxed{\frac{4^4e^8}{4!}}(x-2)^4$$

$$\frac{256}{24}e^8 \rightarrow \boxed{\frac{32e^8}{3}}$$

10. A series expansion for function $f(x)=e^{3x}$ is given by

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$e^{3x}=1+(3x)+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!} \rightarrow \boxed{1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}}$$

$\frac{9}{2}x^3$

$\boxed{\text{(A) } 1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\cdots}$

(B) $1+3x+\frac{3x^2}{2!}+\frac{3x^3}{3!}+\cdots$

(C) $1-3x+\frac{9x^2}{2}-\frac{9x^3}{2}+\cdots$

(D) $1-3x+\frac{3x^2}{2!}-\frac{3x^3}{3!}+\cdots$

11. Let $f$ be the function with initial condition $f(0) = 0$ and derivative $f'(x) = e^{3x}$. Write the first four nonzero terms and the general term of the Maclaurin series for $f$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$f'(x) = e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots + \frac{(3x)^n}{n!} \to \frac{3^n x^n}{n!}$

$f(x) = \int 1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3\,dx$

$f(x) = x + \frac{3x^2}{2} + \frac{9}{2}\left(\frac{x^3}{3}\right) + \frac{27}{6}\left(\frac{x^4}{4}\right) + C$

$C = 0$ since $f(0) = 0$

$\boxed{f(x) = x + \frac{3}{2}x^2 + \frac{3}{2}x^3 + \frac{9}{8}x^4 + \dots \frac{3^n x^{n+1}}{(n+1)n!}}$

$\int\left(\frac{3^n}{n!}\right)x^n dx = \left(\frac{3^n}{n!}\right)\cdot\frac{x^{n+1}}{n+1}$

a)

12. Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{6^n}$.

*Ratio Test

$\lim_{n\to\infty}\left|\frac{x^{n+1}}{6^{n+1}}\cdot\frac{6^n}{x^n}\right| < 1$

$\lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x\cdot\cancel{6^n}}{\cancel{x^n}\cdot\cancel{6^n}\cdot 6}\right| < 1$

$\left|\frac{x}{6}\right| < 1$

$|x| < 6$

$|x - 0| < 6$

$|x - c| < r$

$\boxed{*\text{radius} = 6}$

b) Interval of Convergence

$|x| < 6$

$-6 < x < 6$

$\boxed{\text{test } x = -6}$

$\frac{(-1)^{n+1}(-6)^n}{6^n} \to \frac{(-1)^1(-1)^n(-6)^n}{6^n} \to \frac{(-1)(\cancel{6^n})}{\cancel{6^n}} \to \sum_{n=1}^{\infty}(-1)$ (diverges)

$\boxed{\text{test } x = 6}$

$\frac{(-1)^{n+1}\cancel{(6)^n}}{\cancel{6^n}} \to \sum_{n=1}^{\infty}(-1)^{n+1}$ (diverges)

$\boxed{\text{IOC: } -6 < x < 6}$

## Answers to End of Unit 10 WS #2

| | | | |
|---|---|---|---|
| 1. $\frac{1}{8}$ | 2. D | 3. $f(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3$ | |
| 4. 4 | 5. D | 6. D | 7. $7.2 \times 10^{-7}$ |
| 8. $0 < x \le 6$ | 9. $\frac{32e^8}{3}$ | 10. A | |
| 11. $f(x) = x + \frac{3}{2}x^2 + \frac{3}{2}x^3 + \frac{9}{8}x^4 + \dots + \frac{3^n x^{n+1}}{(n+1)n!}$ | 12. 6 | | |

Name: WS #3 KEY Date: ____________ Period: _____ 9

## BC Calculus Unit 10.5b-10.10 Infinite Series Test Review WS #3

Calculators Allowed:

**10.5b Alternating Series Error Bound**

1. If the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n+1}$ is approximated by the partial sum with 50 terms, what is the alternating series error bound?

$|S-S_n| \le |a_{n+1}|$

$|S-S_{50}| \le |a_{51}|$

$\left|\frac{1}{2(51)+1}\right| = \boxed{\frac{1}{103}}$

2. Approximate an interval for the sum of the convergent alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n 2}{n^2}$ using the Alternating Series Error Bound the first 6 terms.

*calculator (math→0) $\sum_{x=1}^{6}\frac{(-1)^x \cdot 2}{x^2} = -1.62167$ $S_6 \approx -1.62167$

$|S-S_6| \le |a_7|$

$|a_7| = \left|\frac{2}{7^2}\right| = 0.0408$

Sum is partial Sum ± error bound

$S = S_6 \pm a_7$

$S = -1.622 \pm 0.0408$

$-1.622-0.0408 \le S \le -1.622+0.0408$

$\boxed{-1.662 \le S \le -1.58087}$

3. The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ converges to $S$. Based on the alternating series error bound, what is the least number of terms to guarantee a partial sum that is within 0.02 of $S$?

$|S-S_n| < |a_{n+1}|$

$|a_{n+1}| = \left|\frac{1}{\sqrt{n+1}}\right|$

$\frac{1}{\sqrt{n+1}} < 0.02$

$\frac{1}{\sqrt{n+1}} < \frac{0.02}{1}$

$\frac{1}{n+1} < \frac{(0.02)^2}{1}$

$\frac{1}{n+1} < \frac{0.0004}{1}$

$1 < 0.0004(n+1)$

$\frac{1}{0.0004} < n+1$

$2500 < n+1$

$2499 < n$

$n > 2499$

Since $n > 2499$, then the least value for $n = 2500$

4. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{5}{n}$ is approximated by $S_k = \sum_{n=1}^{k}(-1)^{n+1}\frac{5}{n}$, what is the least value of $k$ for which the alternating series error bound guarantees that $|S - S_k| < 0.001$?

$|S-S_k| < |a_{k+1}|$

$|a_{k+1}| = \left|\frac{5}{k+1}\right|$

$\frac{5}{k+1} < 0.001$

$\frac{5}{k+1} < \frac{1}{1000}$

$5000 < k+1$

$4999 < k$ or $k > 4999$

The least value of $k$ is 5000

(A) 999 (B) 1000 (C) 4999 $\boxed{\text{(D) } 5000}$

5. Determine the least number of terms necessary to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n 3}{4^n}$ with an error less than $10^{-3}$.

$|S-S_n| \le |a_{n+1}|$

$|a_{n+1}| = \left|\frac{3}{4^{n+1}}\right|$

$\frac{3}{4^{n+1}} < 10^{-3}$

$\frac{3}{4^{n+1}} < \frac{1}{1000}$

$3000 < 4^{n+1}$

$4^{n+1} > 3000$

$\ln 4^{n+1} > \ln(3000)$

$(n+1)\ln 4 > \ln 3000$

$n+1 > \frac{\ln 3000}{\ln 4}$

$n+1 > 5.7754$

$n > 4.7754$

The least number of terms is $n=5$

**10.10a Finding Taylor Polynomial Approximations of Functions**

6) Find the third-degree Taylor Polynomial for $f(x) = e^{2x}$ about $x = 1$.

$* \; P_n(x) = \sum_{n=0}^{n} \frac{f^n(c)}{n!}(x-c)^n$

$f(x) = e^{2x} \to f(1) = e^2$

$f'(x) = e^{2x}\cdot 2 \to f'(1) = 2e^2$

$f''(x) = 4e^{2x} \to f''(1) = 4e^2$

$f'''(x) = 8e^{2x} \to f'''(1) = 8e^2$

$P_3(x) = e^2 + 2e^2(x-1) + \frac{4e^2}{2!}(x-1)^2 + \frac{8e^2}{3!}(x-1)^3$

$\boxed{P_3(x) = e^2 + 2e^2(x-1) + 2e^2(x-1)^2 + \frac{4}{3}e^2(x-1)^3}$

7) Let $f$ be the function with third derivative $f'''(x) = 12x^{-3}$. What is the coefficient of $(x-1)^4$ in the fourth-degree Taylor polynomial of $f$ about $x = 1$? $\boxed{c=1}$

$* \; \frac{f^n(c)}{n!}(x-c)^n \to \frac{f^4(1)}{4!}(x-1)^4$

$f'''(x) = 12x^{-3}$

$f^4(x) = -36x^{-4} \to f^4(1) = \frac{-36}{1^4} = -36$

$\boxed{\frac{-36}{4!}}(x-1)^4$

$\frac{-36}{4\cdot3\cdot2\cdot1} \to -\frac{\cancel{36}^{\,3}}{\cancel{12}\cdot 2} \to \boxed{\frac{-3}{2}}$

$$* \; P_n(x) = \sum_{n=0}^{n} \frac{f^n(c)}{n!}(x-c)^n$$

8) The function $f$ has derivatives of all orders for all real numbers with $f(4) = 1$, $f'(4) = 3$, $f''(4) = 5$, and $f'''(4) = 12$. Using a third-degree Taylor Polynomial for $f$ about $x = 4$, what is the approximation of $f(4.1)$?

$c=4$

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$P_3(x) = 1 + 3(x-4) + \frac{5}{2}(x-4)^2 + \frac{12}{3!}(x-4)^3$$

$$f(4.1) \approx P_3(4.1) = 1 + 3(4.1-4) + \frac{5}{2}(4.1-4)^2 + 2(4.1-4)^3 = \boxed{1.327}$$

9) The third-degree Taylor Polynomial for a function $f$ about $x = 0$ is $\frac{x^3}{128} - \frac{x^2}{16} + \frac{x}{8} + 4$. What is the value of $f'''(0)$?

$$f(x) = \frac{1}{128}x^3 - \frac{1}{16}x^2 + \frac{1}{8}x + 4$$

$$f'(x) = \frac{3}{128}x^2 - \frac{2}{16}x + \frac{1}{8}$$

$$f''(x) = \frac{6}{128}x - \frac{2}{16}$$

$$f'''(x) = \frac{6}{128}$$

$$\boxed{f'''(0) = \frac{6}{128} \text{ or } \frac{3}{64}}$$

10) Which of the following polynomial approximations is the best for $\sin 2x$ near $x = 0$?

$$* \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots$$

$$\sin(2x) \approx 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!}$$

$$\approx \boxed{2x - \frac{4}{3}x^3 \dots}$$

(A) $2x - 8x^3$ (B) $2 - \frac{4}{3}x^2$ $\boxed{\text{(C) } 2x - \frac{4}{3}x^3}$ (D) $2 - \frac{4}{3}x$

$$* \; R_n(x) \leq \left| \frac{\max \left| f^{(n+1)}(z) \right|}{(n+1)!} (x-c)^{n+1} \right|$$

**10.10b Lagrange Error Bound**

11) The fourth-degree Maclaurin polynomial for $\cos x$ is given by $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. Use the Lagrange error bound to estimate the error in using this polynomial to approximate $\cos\frac{\pi}{3}$.

$c=0$

$$R_4(\pi/3) \leq \left| \frac{(\max) f^{(5)}(z)}{5!} (\pi/3 - 0)^5 \right| \longrightarrow \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 = 0.0105$$

$$\boxed{R_4(\pi/3) \leq 0.0105}$$

$$f^5(x) = -\sin x$$

Since $-1 < \sin x < 1$,

the max value of $\sin x = 1$

12) The function $f$ has derivatives of all orders for all real numbers and $f^{(4)}(x) = e^{\sin x}$. If the third-degree Taylor Polynomial for $f$ about $\boxed{x = 0}$ is used to approximate $f$ on [0,1], what is the Lagrange error bound for the maximum error on [0,1]?

$c = 0$

$$R_3(x) \le \left| \frac{\max f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \right|$$

$$R_3(x) \le \left| \frac{\max f^4(z)}{4!}(x-c)^4 \right|$$

$f^4(x) = e^{\sin x}$

$f^4(1) = e^{\sin 1}$

* The greatest value of $\sin x$ on [0,1] is $\boxed{\sin(1)}$ since $\sin(1) > \sin(0)$
* $c = 0$
* $x = 1$

$$R_3(1) \le \left| \frac{e^{\sin(1)}}{4!}(1-0)^4 \right| = \boxed{0.0967}$$

13) Assume a third-degree Taylor Polynomial about $x = 2$ is used for the approximation $f$ and $|f^{(4)}(x)| \le 12$ for all $x \ge 1$. Which of the following represents the Lagrange error bound in the approximation of $f(2.5)$?

$\boxed{c = 2}$

$$R_3(x) \le \left| \frac{\max f^4(x)}{4!}(x-c)^4 \right|$$

* $\max |f^4(x)| = 12$
* $c = 2$
* $x = 2.5$

$$R_3(2.5) \le \left| \frac{12}{4!}(2.5-2)^4 \right| = 0.03125 = \boxed{\frac{1}{32}}$$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{16}$ $\boxed{\text{(D) } \frac{1}{32}}$

14) Determine the degree of the Taylor Polynomial about $x = 0$ for $f(x) = e^x$ required for the error in the approximation of $f(0.8)$ to be less than 0.005.

$$R_n(x) \le \left| \frac{\max f^{(n+1)}(x)}{(n+1)!}(x-c)^{n+1} \right| < 0.005$$

* $c = 0$
* $x = 0.8$

* $f^n(x) = e^x \rightarrow$ max value of $e^x$ on [0,0.8] is $e^{0.8}$ since $e^{0.8} > e^0$

$$\left| \frac{e^{0.8}}{(n+1)!}(0.8-0)^{n+1} \right| < 0.005$$

* test n-values using calculator.
* when $n = 5$,

$$\frac{e^{0.8}}{6!}(0.8)^6 = 8.10 \times 10^{-4} = 0.00081 < 0.005$$

$\boxed{\text{So } n = 5 \text{ (5}^{th}\text{ degree)}}$

15)

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ | $f^{(4)}(x)$ |
|---|---|---|---|---|---|
| 2 | 112 | 164 | 214 | 312 | 345 |

Let $f$ be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of $f$ are given for $x = 2$. Use the Lagrange error bound to show that the third-degree Taylor Polynomial for $f$ about $x = 2$ approximates $f(1.9)$ with an error less than 0.002.

$$R_n(x) \le \left| \frac{\max f^{n+1}(x)}{(n+1)!}(x-c)^{n+1} \right|$$

* $n=3$
* $c=2$
* $x=1.9$

$$R_3(1.9) \le \left| \frac{\max f^4(x)}{4!}(1.9-2)^4 \right| = \frac{345}{4!}(-0.1)^4$$

$$R_3(1.9) \le \boxed{0.001438 < 0.002}$$

## 10.8a Radius and Interval of Convergence of Power Series

(I.O.C.)

$\boxed{\text{*Remember to test endpoints!!}}$

**Find the interval of convergence for each power series.**

16) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x+4)^n}{n}$

*Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x+4)^{n+1}}{(n+1)} \cdot \frac{n}{(x+4)^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{(x+4)^1 \cancel{(x+4)^n}}{\cancel{(x+4)^n}} \cdot \frac{n}{n+1} \right| < 1$$

$$|x+4| < 1$$

$$-1 < x+4 < 1$$

$$-5 < x < -3$$

*test $x=-5$:

$$\frac{(-1)^n(-5+4)^n}{n} \to \frac{(1)^n}{n}$$

diverges

test $x=-3$:

$$\frac{(-1)^n(-3+4)^n}{n} \to \frac{(-1)^n}{n}$$

converges (AST)

$$\boxed{\begin{array}{c}\text{I.O.C.}\\ -5 < x \le -3\end{array}}$$

17) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, n!\, (x-4)^n}{3^n}$

*Ratio Test

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-4)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!(x-4)^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(x-4)^n}(x-4)\cdot\cancel{3^n}}{\cancel{(x-4)^n}\cdot\cancel{3^n}\cdot 3} \cdot \frac{(n+1)\cancel{n!}}{\cancel{n!}} \right| < 1$$

* Since $\lim_{n\to\infty}(n+1) = \infty$, the series converge only at its center $(x=c)$, $\boxed{x=4}$

18) What is the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2^n}$?

* $|x-c| < r$

c = center, r = radius

*Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(x+2)^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(x+2)^n}(x+2)\cdot\cancel{2^n}}{\cancel{(x+2)^n}\cdot\cancel{2^n}\cdot 2} \right| < 1$$

$$\left| \frac{x+2}{2} \right| < 1$$

$$|x+2| < 2$$

$$\downarrow$$

$$|x-c| < r$$

center $c=-2$

$\boxed{\text{radius } r = 2}$

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19) What is the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{n}{n+1}(-kx)^{n-1}$, where $k$ is a positive integer?

*treat k like you would a constant or coefficient like 2 or 3

*Ratio Test

$$\lim_{n\to\infty}\left|\frac{(n+1)\cdot(-kx)^{n}}{n+2}\cdot\frac{n+1}{n(-kx)^{n-1}}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(-kx)^{n}}}{\cancel{(-kx)^{n}}(-kx)^{-1}}\cdot\frac{(n+1)(n+1)}{n(n+2)}\right|<1$$

$$\left|\frac{1}{(-kx)^{-1}}\right|<1$$

$$|(-kx)^{1}|<1$$

$$|kx|<1$$

$$-1<kx<1$$

$$\frac{-1}{k}<\frac{kx}{k}<\frac{1}{k}$$

$$\boxed{\frac{-1}{k}<x<\frac{1}{k}}$$

test endpoints

$\boxed{\text{test } x=\frac{-1}{k}}$

$$\lim_{n\to\infty}\frac{n}{n+1}\cdot\left(-k\cdot\frac{-1}{k}\right)^{n-1}$$

$$\lim_{n\to\infty}\frac{n}{n+1}(1)^{n-1}\neq 0$$

(diverges)

$\boxed{\text{test } x=1/k}$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}(-1)^{n-1}\right)\neq 0,$$

diverges

20) If the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges at $x=7$ and diverges at $x=8$, which of the following must be true?

scenario 1 may fail → ~~I.~~ The series converges at $x=1$.

scenario 1 and 2 passes → $\boxed{\text{II.}}$ The series converges at $x=2$.

scenario 1 fails, scenario 2 may fail → ~~III.~~ The series diverges at $x=0$.

*center at $x=4$

$\boxed{\text{Scenario 1}}$

r=3

1 4 7

$$|x-4|<3$$

$$-3<x-4<3$$

$$1<x\leq 7$$

$\boxed{\text{OR}}$

$\boxed{\text{Scenario 2}}$

r=4

0 4 8

$$|x-4|<4$$

$$-4<x-4<4$$

$$0<x<8$$

(A) I only $\boxed{\text{(B) II only}}$ (C) I and II only (D) II and III only

**10.9 Finding Taylor or Maclaurin Series for a Function**

21) What is the coefficient of $x^6$ in the Taylor Series about $x=0$ for the function $f(x)=\frac{e^{3x^2}}{4}$?

$$e^{x}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}$$

$$e^{3x^2}=1+(3x^2)+\frac{(3x^2)^2}{2}+\frac{(3x^2)^3}{3!}$$

$$\frac{1}{4}\cdot e^{3x^2}=\frac{1}{4}\left(1+3x^2+\frac{9x^4}{2}+\frac{27x^6}{6}\right)$$

$$\frac{1}{4}e^{3x^2}\approx\frac{1}{4}+\frac{3}{4}x^2+\frac{9}{8}x^4+\boxed{\frac{27}{24}}x^6$$

coefficient is:

$\frac{27}{24}$ or $\boxed{\frac{9}{8}}$

22) Write the first four non-zero terms for the Taylor Series for the function $f(x)=2x\cos x$ about $x=0$?

$$*\ \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+\frac{(-1)^n x^{2n}}{(2n)!}$$

$$2x\cdot\cos x\approx 2x\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\right]=2x-\frac{2x^3}{2}+\frac{2x^5}{24}-\frac{2x^7}{720}$$

$$=\boxed{2x-x^3+\frac{x^5}{12}-\frac{x^7}{360}}$$

23) What is the sum of the series $1 - \frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \cdots + \frac{(-1)^n 3^{2n}}{(2n)!}$?

(A) $\ln 3$ (B) $e^3$ (C) $\sin 3$ (D) $\boxed{\cos 3}$

* the form and pattern of the series looks like that of cos x

$*\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!}$

$$\boxed{\cos 3 = 1 - \frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \ldots + \frac{(-1)^n (3)^{2n}}{(2n)!}}$$

24) Write the first four non-zero terms in the Maclaurin Series for the function $f(x) = x \sin 2x$.

$*\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$x \cdot \sin x = x\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots +\right] \ldots \frac{(-1)^n \cdot x^{2n+1} \cdot x}{(2n+1)!}$

$x \cdot \sin(2x) = x\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots\right] + \ldots \frac{(-1)^n (2x)^{2n+1} x}{(2n+1)!}$

$$\boxed{x\sin(2x) \approx 2x^2 - \frac{2^3 x^4}{3!} + \frac{2^5 x^6}{5!} - \frac{2^7 x^8}{7!}}$$

$$\boxed{\text{*General term is: } \frac{(-1)^n (2)^{2n+1} x^{2n+2}}{(2n+1)!}}$$

25) Which of the following is the Maclaurin Series for the function $f$ defined by $f(x) = 1 + x^2 + \cos x$?

(A) $\boxed{2 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots}$ (B) $2 + \frac{3x^2}{2} + \frac{x^4}{24} + \cdots$ (C) $1 + x + x^2 - \frac{x^3}{6} + \cdots$ (D) $2 + x + \frac{3x^2}{2} + \frac{x^3}{6} + \cdots$

$*\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \frac{(-1)^n x^{2n}}{(2n)!}$

$f(x) = 1 + x^2 + \cos x$

$f(x) = 1 + x^2 + 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$f(x) = 1 + 1 + x^2 - \frac{1}{2}x^2 + \frac{x^4}{24} + \ldots$

$$\boxed{f(x) = 2 + \frac{1}{2}x^2 + \frac{x^4}{24} + \ldots}$$

**10.8b Representing Functions as a Power Series**

26) What is the coefficient of $x^5$ in the Taylor series for the function $f(x) = e^x \sin x$ about $x = 0$?

$*e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$ | $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$(e^x)(\sin x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)$

$= \ldots + \frac{x^5}{5!} - \frac{x^5}{2!3!} + \frac{x^5}{4!}$

$\frac{x^5}{120} - \frac{x^5}{12} + \frac{x^5}{24}$

$\frac{1x^5}{120} - \frac{10x^5}{120} + \frac{5x^5}{120} = \frac{-4}{120}x^5$

coefficient is $\frac{-4}{120}$ or $\boxed{\frac{-1}{30}}$

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27) If the function $f$ is defined by $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$, then $\boxed{f'(x) = ?}$ Write the first four nonzero terms and the general term of the Taylor series about $x = 0$.

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$f(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots \frac{x^{2n}}{n!}$$

$$f'(x) = 0 + 2x + \frac{1}{2!}\cdot 4x^3 + \frac{1}{3!}\cdot 6x^5 + \frac{1}{4!}\cdot 8x^7 + \ldots \frac{2n}{n!}x^{2n-1}$$

$$\boxed{f'(x) = 2x + 2x^3 + x^5 + \frac{x^7}{3} + \ldots + \frac{2nx^{2n-1}}{n!}}$$

28) Let $f$ be the function defined by $f(x) = e^{3x}$. Find the Maclaurin series for the derivative $f'$. Write the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!}$$

$$\boxed{f'(x) = 3 + 9x + \frac{27}{2}x^2 + \frac{27}{2}x^3 + \ldots + \frac{3^n \cdot n x^{n-1}}{n!}}$$

$$f(x) = e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \ldots + \frac{(3x)^n}{n!}$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \ldots + \frac{3^n x^n}{n!}$$

$$f'(x) = 0 + 3 + \frac{9}{2}\cdot 2x + \frac{27}{6}\cdot 3x^2 + \frac{81}{24}\cdot 4x^3 + \ldots \frac{3^n}{n!}\cdot n x^{n-1}$$

* If $f(x)$ general term is left as $f(x) = \frac{(3x)^n}{n!} \rightarrow f'(x) = \frac{n(3x)^{n-1}(3)}{n!}$

general term can also be $\frac{3n(3x)^{n-1}}{n!}$

29) Find the third-degree Taylor Polynomial for $f(x) = \sin x \cos x$ about $x = 0$.

* $\sin x = x - \frac{x^3}{3!} + \ldots$ * $\cos x = 1 - \frac{x^2}{2!} + \ldots$

$$f(x) = (\sin x)(\cos x) = \left(x - \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right)$$

$$P_3(x) = x - \frac{x^3}{2} - \frac{x^3}{6} + \cancel{\frac{x^5}{12}}$$

$$= x - \frac{3x^3}{6} - \frac{1x^3}{6}$$

$$P_3(x) = x - \frac{4x^3}{6}$$

$$\boxed{P_3(x) = x - \frac{2}{3}x^3}$$

30) If $f'(x) = \frac{4}{1+x}$ and $f(0) = 0$, write the first four nonzero terms and the general term of the Maclaurin series for $f(x)$.

$$f'(x) = \frac{4}{1-(-x)} \quad \frac{a_1}{1-r} \longrightarrow \underset{n=0}{4} - \underset{n=1}{4x} + \underset{n=2}{4x^2} - \underset{n=3}{4x^3} + \ldots + (-1)^n 4x^n$$

$$\boxed{f(x) = 4x - 2x^2 + \frac{4}{3}x^3 - x^4 + \ldots \frac{(-1)^n 4x^{n+1}}{n+1}}$$

$$f(x) = \int 4 - 4x + 4x^2 - 4x^3\,dx$$

$$f(x) = 4x - \frac{4x^2}{2} + \frac{4x^3}{3} - \frac{4x^4}{4} + C$$

plug in (0,0) solve for C

$0 = 0 + 0 + \ldots + C$

$C = 0$