Name: WS #1 KEY Date:__________ Period:_____

End-of-Unit 10 Review – Infinite Sequences and Series

Lessons 10.~~5~~6 through 10.10

1. If the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n^3 - 1}$ is approximated by $S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{2n^3 - 1}$, what is the least value of k for which the alternating series error bound guarantees that $|S - S_k| < 10^{-3}$? $\rightarrow \frac{1}{10^3} \rightarrow \frac{1}{1000}$

$|S - S_k| \le |a_{k+1}| < \frac{1}{1000}$

$\frac{1}{2(k+1)^3 - 1} < \frac{1}{1000}$

$1000 < 2(k+1)^3 - 1$

$1001 < 2(k+1)^3$

$500.5 < (k+1)^3$

$7.939 < k+1$

$6.939 < k$

$\boxed{k > 6.939}$

(A) 6 $\boxed{\text{(B) } 7}$ (C) 8 (D) 9

2. If the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4n + 1}$ is approximated by the partial sum with 15 terms, what is the alternating series error bound?

$\underbrace{\text{series error bound}}_{\ast\text{next term } a_{n+1}}$

16^{th} term is $\frac{1}{4(16)+1} = \boxed{\frac{1}{65}}$

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{61}$ $\boxed{\text{(D) } \frac{1}{65}}$

3. Let $P(x) = 3 - 2x^2 + 5x^4$ be the fourth-degree Taylor Polynomial for the function f about $x = 0$. What is the value of $f^{(4)}(0)$?

$\ast P_n(x) = \frac{f^n(c)}{n!}(x-c)^n$ | $n = 4$, $c = 0$, $x = 0$

$f^4(x) = \frac{f^4(0)}{4!}(x-0)^4 = 5x^4$

$\frac{f^4(0)}{4!} = 5$

$f^4(0) = 5 \cdot 4! = \boxed{120}$

4. The function f has derivatives of all orders for all real numbers with $f(2) = -2$, $f'(2) = 4$, $f''(2) = 8$, and $f'''(2) = 14$. Using the third-degree Taylor Polynomial for f about $x = 2$, what is the approximation of $f(2.2)$?

$c = 2$

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$P_3(x) = -2 + 4(x-2) + \frac{8}{2!}(x-2)^2 + \frac{14}{3!}(x-2)^3$

$f(2.2) \approx -2 + 4(2.2-2) + 4(2.2-2)^2 + \frac{7}{3}(2.2-2)^3 = \boxed{-1.021}$

$*$LaGrange $R_n(x) \leq \frac{|\max(f^{n+1}(z))|}{(n+1)!}(x-c)^{n+1}$

$c=0,\ x=1$

5. Let f be a function that has derivatives of all orders for all real numbers and let $P_4(x)$ be the fourth-degree Taylor Polynomial for f about $x = 0$. $|f^{(n)}(x)| \leq \frac{n}{n+1}$, for $1 \leq n \leq 6$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_4(1)| \leq k$?

$$R_4(x) \leq \left|\frac{\max(f^5(z))}{5!}(1-0)^5\right| \leq k$$

$$\leq \frac{\left(\frac{5}{6}\right)}{5!}(1-0)^5 \rightarrow \frac{5}{6}\cdot\frac{1}{5!} \rightarrow \boxed{\frac{1}{6\cdot 4!} \leq k}$$

(A) $\frac{4}{5}\left(\frac{1}{4!}\right)$ (B) $\frac{4}{5}\left(\frac{1}{5!}\right)$ $\boxed{\text{(C) } \frac{1}{6}\left(\frac{1}{4!}\right)}$ (D) $\frac{1}{6}\left(\frac{1}{5!}\right)$

6. The third Maclaurin polynomial for $\sin x$ is given by $f(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.3)$, what is the Lagrange error bound?

$*\ \frac{\max[f^{n+1}(z)]}{(n+1)!}(x-c)^{n+1}$

$*\ c=0$

$*\ x=0.3$

$*\ -1 \leq \sin x \leq 1$

$$\boxed{R_3(0.3) \leq \frac{(1)}{4!}(0.3-0)^4 = 3.375\times10^{-4}}$$

7. Find the interval of convergence for the power series $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$.

$*$Apply Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+2}}{(n+2)}\cdot\frac{n+1}{(x-1)^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x-1)^n}\cdot(x-1)^2}{\cancel{(x-1)^n}\cdot(x-1)}\cdot\frac{n+1}{n+2}\right| < 1$$

$$\lim_{n\to\infty}|x-1| < 1$$

$-1 < x-1 < 1$

$0 < x < 2$

$*$ Test endpoints!

test $x=0$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(0-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}\frac{(1)^{n+1}}{n+1} \text{ diverges}$$

test $x=2$:

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2-1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1} \text{ converges (AST)}$$

$$\boxed{0 < x \leq 2}$$

8. If the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{5^n}$ is 5, what is the interval of convergence?

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-0)^n}{5^n}$$

$c=0$

$r=5$

$|x-c| < r$

$|x-0| < 5$

$|x| < 5$

$-5 < x < 5$

$*$test endpoints:

$x=-5$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-5)^n}{5^n} \rightarrow \frac{(5)^n}{5^n} \text{ diverges}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(5)^n}{5^n} \rightarrow (-1)^n \text{ diverges}$$

test $x=5$

$$\boxed{-5 < x < 5}$$

$* \; e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$

$* \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \frac{x^{2n-1}}{(2n-1)!}$

9. Which of the following is an expression for a function f that has the Maclaurin Series $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$?

series alternate signs

~~(A)~~ $\cos x$ ~~(B)~~ $e^x - \sin x$ $\boxed{\text{(C) } \frac{1}{2}(e^x + e^{-x})}$ ~~(D)~~ $e^{x^2} = 1 + (x^2) + \frac{x^4}{2!} + \ldots$

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

$$1 + \frac{x^2}{2!} + \underbrace{2\left(\frac{x^3}{3!}\right)}_{\text{does not match series}}$$

$$\frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right)\right]$$

$$\frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} \ldots\right]$$

$$= \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots} \checkmark$$

10. Find the Maclaurin Series for the function $f(x) = 2\sin x^3$. Write the first four non-zero terms.

$* \; \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$$2\sin(x^3) = 2\left[x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!}\right] = \boxed{2x^3 - \frac{2x^9}{3!} + \frac{2x^{15}}{5!} - \frac{2x^{21}}{7!}}$$

11. It is known the Maclaurin series for the function $\frac{1}{1+x}$ is defined by $\sum_{n=0}^{\infty}(-1)^n x^n$. Use this fact to find the first four nonzero terms and the general term for the power series expansion for $\frac{x^2}{1+x^2}$.

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots + (-1)^n(x)^n$$

$$\frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n(x)^{2n}$$

$$x^2 \cdot \frac{1}{1-(-x^2)} = x^2 - x^4 + x^6 - x^8 + \ldots \quad (-1)^n x^{2n} \cdot x^2 \rightarrow \boxed{(-1)^n x^{2n+2}}$$

12. Let $T(x) = 7 - 3(x-3) + 5(x-3)^2 - 2(x-3)^3 + 6(x-3)^4$ be the fourth-degree Taylor Polynomial for the function f about $x = 3$. Find the third-degree Taylor Polynomial for the derivative f' about $x = 3$ and use it to approximate $f'(3.3)$.

$$T_3'(x) = -3(1) + 10(x-3) - 6(x-3)^2 + 24(x-3)^3$$

$$f'(3.3) \approx T_3'(3.3) = -3 + 10(3.3-3) - 6(0.3)^2 + 24(0.3)^3 = \boxed{0.108}$$

Taylor Polynomial is a polynomial that will approximate other function's values in a region that is nearby the "center"
*a tangent line is essentially a first degree taylor polynomial.

n^{th} degree taylor polynomial:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Alternating Series Remainder:
Suppose an alternating series converges by AST. If the Series has Sum S, then $|R_n| = |S - S_n| \le |a_{n+1}|$

*This means that the **maximum error** for the n^{th} term partial Sum S_n is no greater than the absolute value of the first unused term a_{n+1}

Taylor Series: A General method for writing a power series representation for a function.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f^{(n)}$ represents the n^{th} derivative evaluated at f.

Power Series: Written in form $\sum_{n=0}^{\infty} a_n (x-c)^n$ where c and a_n (coefficients) are numbers:
*Taylor and Maclaurin series are special cases of power series
For a power series centered at c, precisely one of the following is true:
1) The series converges only at c (ALL power series converge at least at their center) (Radius of convergence = 0)
2) The series converges for all x (function and infinite series have exact same values everywhere) → Radius = ∞
3) The series converges within a certain Radius of Convergence such that series converges for $|x - c| < R$
→The **interval of Convergence** (I.O.C.) is **[(c – R , c + R)]**
*Be sure to TEST convergence of endpoints
*Typically, you want to use the **RATIO TEST** to determine Radius of Convergence

Maclaurin Series : is the special case of Taylor series when **c = 0.**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Special Maclaurin Series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} \quad \text{IOC: All Reals}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} \quad \text{IOC: All Reals}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!} \quad \text{IOC: All Reals}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)} \quad \text{IOC}: -1 \le x \le 1$$

Geometric Series below based on

$$S = \frac{a_1}{1-r} \quad \text{IOC: } -1 < x < 1$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots (-1)^n x^n + \dots$$

IOC: $-1 < x < 1$

$$\frac{1}{x} = \frac{1}{1-[-(x-1)]} = 1 - (x-1) + (x-1)^2 - \dots (-1)^n (x-1)^n$$

IOC: $0 < x < 2$

$$\ln x = \int \frac{1}{x}\,dx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \dots \frac{(-1)^{n-1}(x-1)^n}{n}$$

IOC: $0 < x \le 2$

LaGrange Error Bound *This is similar to the Alternating Series Remainder. However, this method offers a way to determine the maximum error (remainder) when we do a Taylor polynomial approximation using a certain number of terms for a specific function.

$$R_n(x) = \left| \frac{|f^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} \right| \le \left| \frac{\max|f^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} \right|$$

* The remainder for an n^{th} degree polynomial is found by taking the $(n+1)^{st}$ (**first unused**) derivative at "z" *We are not expected to find the exact value of z. (If we could, then an approximation would not be necessary) *We want to maximize the $(n+1)^{st}$ derivative on the interval from [x , c] in order to find a safe upper bound for the $|f^{(n+1)}(z)|$ *The maximum error bound is the worst case scenario for the interval in which our actual approximation can live. **College Board will provide strictly increasing and decreasing functions. (So we only have to choose between f(c) and f(x) (the endpoints). This will allow us to determine the max value much more accurately.

Alternating Series Remainder:
Suppose an alternating series converges by AST
(such that $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing), then

$|R_n| = |S - S_n| \le |a_{n+1}|$

*This means that the **maximum error** for the n^{th} term partial Sum S_n is no greater than the absolute value of the first unused term a_{n+1}