Name: WS #2 KEY Date:______________

End of Unit 10 WS#2 Infinite Sequences and Series

$\dfrac{f^n(c)}{n!}(x-c)^n$

1. Let f be the function defined by $f(x) = \underbrace{3x}\,\underbrace{\cos x}$. What is the coefficient of x^5 in the Taylor Series for f about $x = 0$? $(c=0)$

*product Rule

$f'(x) = 3\cos x + -3x\sin x$

$f''(x) = -3\sin x - 3\sin x - 3x\cos x$

$= -6\sin x - 3x\cos x$

$f'''(x) = -6\cos x - 3\cos x + 3x\sin x$

$f'''(x) = -9\cos x + 3x\sin x$

$f^4(x) = 9\sin x + 3\sin x + 3x\cos x$

$= 12\sin x + 3x\cos x$

$f^5(x) = 12\cos x + 3\cos x - 3x\sin x$

$f^5(x) = 15\cos x - 3x\sin x$

$f^5(0) = 15\cos 0 - 3(0)\sin(0) = \boxed{15}$

5th degree term is

$\dfrac{f^5(0)}{5!}(x-0)^5$

$\boxed{\dfrac{15}{5!}}x^5$

$\to \dfrac{3\cdot 5}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \boxed{\dfrac{1}{8}}$

2. Determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ with an error less than 0.0001.

$* \; |S - S_n| \le |a_{n+1}|$

*what makes $\left|\dfrac{1}{(n+1)^4}\right| < 0.0001$

$\dfrac{1}{(n+1)^4} < \dfrac{1}{10000}$

$(n+1)^4 > 10000$

$n+1 > \sqrt[4]{10000}$

$n+1 > 10$

$\boxed{n > 9}$

(A) 7 (B) 8 (C) 9 $\boxed{\text{(D) } 10}$

3. Find the third-degree Taylor Polynomial for the function $f(x) = \sqrt{x}$ about $x = 2$.

$* \; \dfrac{f^n(c)}{n!}(x-c)^n \to P_3(x) = \sqrt{2} + \dfrac{\sqrt{2}}{4}(x-2) - \dfrac{\frac{\sqrt{2}}{16}}{2!}(x-2)^2 + \dfrac{\frac{3\sqrt{2}}{64}}{3!}(x-2)^3$

$\to \dfrac{\sqrt{2}}{16}\cdot\dfrac{1}{2} = \dfrac{\sqrt{2}}{32}$ $\to \dfrac{\cancel{3}\sqrt{2}}{64}\cdot\dfrac{1}{\cancel{3}\cdot 2} \to \dfrac{\sqrt{2}}{128}$

$f(x) = x^{1/2} \to f(2) = \sqrt{2}$

$f'(x) = \frac{1}{2}x^{-1/2} \to f'(x) = \dfrac{1}{2\sqrt{x}} \to f'(2) = \dfrac{1}{2\sqrt{2}}$ or $\dfrac{\sqrt{2}}{4}$

$f''(x) = \frac{-1}{4}x^{-3/2} \to f''(x) = \dfrac{-1}{4x^{3/2}} \to f''(2) = \dfrac{-1}{4(\sqrt{2})^3} = \dfrac{-1}{8\sqrt{2}} \to \dfrac{-\sqrt{2}}{16}$

$f'''(x) = \frac{3}{8}x^{-5/2} \to \dfrac{3}{8x^{5/2}} \to f'''(2) = \dfrac{3}{8(\sqrt{2})^5} \to \dfrac{3}{32\sqrt{2}} \to \dfrac{3\sqrt{2}}{64}$

$\boxed{P_3(x) = \sqrt{2} + \dfrac{\sqrt{2}}{4}(x-2) - \dfrac{\sqrt{2}}{32}(x-2)^2 + \dfrac{\sqrt{2}}{128}(x-2)^3}$

4. What is the coefficient of x^3 in the Maclaurin series for the function $\left(\frac{1}{1-x}\right)^2$?

$f(x) = \dfrac{1}{(1-x)^2} = (1-x)^{-2}$

$f'(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$

$f''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$

$f'''(x) = -24(1-x)^{-5}(-1) = 24(1-x)^{-5}$

$f'''(0) = 24(1-0)^{-5} = \dfrac{24}{(1)^5} = 24$

$\dfrac{f^3(0)}{3!}(x-0)^3 \to \boxed{\dfrac{24}{3!}}x^3$

$\to \dfrac{24}{3\cdot 2} = \dfrac{24}{6} = \boxed{4}$

5. The function f has derivatives of all orders for all real numbers and $f^{(4)}(x) \leq \frac{1}{2}$. If a third-degree Taylor Polynomial for f about $x = 0$ is used to approximate f on $[0,1]$. What is the Lagrange error bound for the maximum error on interval $[0,1]$ in the approximation of $f(1)$?

$c = 0$, $x = 1$

*LaGrange Error Bound

$$R_n(x) \leq \frac{|\max(f^{n+1}(z))|}{(n+1)!}(x-c)^{n+1}$$

$$R_3(x) \leq \frac{|\max(f^4(x))|}{(3+1)!}(1-0)^4$$

$$R_3(1) \leq \frac{\left(\frac{1}{2}\right)}{4!}(1)^4 = \frac{1}{2}\cdot\frac{1}{4!} \rightarrow \frac{1}{2}\cdot\frac{1}{24} = \boxed{\frac{1}{48}}$$

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{24}$ $\boxed{\text{(D) } \frac{1}{48}}$

6. What is the alternating series error bound, if the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n+2}$ is approximated by the partial sum with 15 terms?

$$|S - S_n| \leq |a_{n+1}| \rightarrow a_{n+1} \rightarrow a_{15+1} \rightarrow \left|\frac{1}{5(15+1)+2}\right| \rightarrow \boxed{\frac{1}{82}}$$

* use the 16^{th} term (a_{16}) to determine the error bound of partial sum (S_{15})

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{77}$ $\boxed{\text{(D) } \frac{1}{82}}$

7. $\max\limits_{0\leq x\leq 2}|f^{(5)}(x)| = 3.6$ $\max\limits_{0\leq x\leq 2}|f^{(6)}(x)| = 8.1$ $\max\limits_{0\leq x\leq 2}|f^{(7)}(x)| = 11.3$

Let $P(x)$ be the fifth-degree Taylor Polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \leq x \leq 2$ is given in the table above. What is the smallest value of k for which the Lagrange error bound guarantees that $|f(0.2) - P(0.2)| \leq k$?

$c = 0$
$x = 0.2$

$$R_5(x) = \frac{|\max f^6(x)|}{6!}(x-c)^6$$

$$R_5(0.2) = \frac{(8.1)}{6!}(0.2-0)^6 = (0.01125)(0.2)^6 = \boxed{7.2 \times 10^{-7}}$$

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8. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-3)^n}{n3^n}$. (I.o.c.)

*Ratio Test

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{(n+1)\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{(x-3)^n}\right|<1$$

$$\lim_{n\to\infty}\left|\frac{\cancel{(x-3)^n}(x-3)\cdot\cancel{3^n}}{\cancel{(x-3)^n}\cdot\cancel{3^n}\cdot 3}\cdot\frac{n}{n+1}\right|<1$$

$$\left|\frac{x-3}{3}\right|<1$$

$|x-c|<r$

$$|x-3|<3$$

$$-3<x-3<3$$

$$0<x<6$$

*test endpoints!

at $x=0$

$$\frac{(-1)^{n+1}(-3)^n}{n\cdot 3^n}\rightarrow\frac{(-1)^n(-3)^n\cdot(-1)^1}{n\cdot 3^n}$$

$(-1)^n(-1)^1(-3)^n$

$$\rightarrow\frac{\cancel{(3)^n}(-1)^1}{n\,\cancel{(3^n)}}\rightarrow\frac{-1}{n}$$

(diverges)

at $x=6$

$$\frac{(-1)^{n+1}\cancel{(3)^n}}{n\cdot\cancel{3^n}}\rightarrow\frac{(-1)^{n+1}}{n}$$ (converges)

Ioc: $\boxed{0<x\le 6}$

9. What is the $\boxed{\text{coefficient}}$ of $(x-2)^4$ in the Taylor Polynomial for $f(x)=e^{4x}$ about $x=2$? $c=2$

$f(x)=e^{4x}$

$f'(x)=e^{4x}(4)$

$f''(x)=4^2e^{4x}$

$f'''(x)=4^3e^{4x}$

$f^4(x)=4^4e^{4x}$

$$*\ P_n(x)=\frac{f^n(c)}{n!}(x-c)^n$$

$$f^4(2)=4^4e^{4(2)}=4^4e^8$$

$$\frac{f^4(2)}{4!}(x-2)^4$$

$$\boxed{\frac{4^4e^8}{4!}}(x-2)^4$$

$$\frac{256}{24}e^8\rightarrow\boxed{\frac{32e^8}{3}}$$

10. A series expansion for function $f(x)=e^{3x}$ is given by

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$e^{3x}=1+(3x)+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}\rightarrow\boxed{1+3x+\frac{9x^2}{2}+\frac{27x^3}{6}}$$

$\frac{9}{2}x^3$

$\boxed{\text{(A) } 1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\cdots}$

(B) $1+3x+\frac{3x^2}{2!}+\frac{3x^3}{3!}+\cdots$

(C) $1-3x+\frac{9x^2}{2}-\frac{9x^3}{2}+\cdots$

(D) $1-3x+\frac{3x^2}{2!}-\frac{3x^3}{3!}+\cdots$

11. Let f be the function with initial condition $f(0) = 0$ and derivative $f'(x) = e^{3x}$. Write the first four nonzero terms and the general term of the Maclaurin series for f.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$f'(x) = e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots + \frac{(3x)^n}{n!} \to \frac{3^n x^n}{n!}$

$f(x) = \int 1 + 3x + \frac{9}{2}x^2 + \frac{27}{6}x^3 \, dx$

$f(x) = x + \frac{3x^2}{2} + \frac{9}{2}\left(\frac{x^3}{3}\right) + \frac{27}{6}\left(\frac{x^4}{4}\right) + C$

$C = 0$ since $f(0) = 0$

$\boxed{f(x) = x + \frac{3}{2}x^2 + \frac{3}{2}x^3 + \frac{9}{8}x^4 + \ldots \frac{3^n x^{n+1}}{(n+1)n!}}$

$\int \left(\frac{3^n}{n!}\right) x^n dx = \left(\frac{3^n}{n!}\right) \cdot \frac{x^{n+1}}{n+1}$

a)

12. Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{6^n}$.

*Ratio Test

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{6^{n+1}} \cdot \frac{6^n}{x^n} \right| < 1$

$\lim_{n\to\infty} \left| \frac{\cancel{x^n} \cdot x \cdot \cancel{6^n}}{\cancel{x^n} \cdot \cancel{6^n} \cdot 6} \right| < 1$

$\left|\frac{x}{6}\right| < 1$

$|x| < 6$

$|x - 0| < 6$

$|x - c| < r$

$\boxed{\text{*radius} = 6}$

b) Interval of Convergence

$|x| < 6$

$-6 < x < 6$

$\boxed{\text{test } x = -6}$

$\frac{(-1)^{n+1}(-6)^n}{6^n} \to \frac{(-1)^1(-1)^n(-6)^n}{6^n} \to \frac{(-1)\cancel{(6)^n}}{\cancel{6^n}} \to \sum_{n=1}^{\infty}(-1)$ (diverges)

$\boxed{\text{test } x = 6}$

$\frac{(-1)^{n+1}\cancel{(6)^n}}{\cancel{6^n}} \to \sum_{n=1}^{\infty}(-1)^{n+1}$ (diverges)

$\boxed{\text{IOC: } -6 < x < 6}$

Answers to End of Unit 10 WS #2

1. $\frac{1}{8}$	2. D	3. $f(x) = \sqrt{2} + \frac{\sqrt{2}}{4}(x-2) - \frac{\sqrt{2}}{32}(x-2)^2 + \frac{\sqrt{2}}{128}(x-2)^3$

4. 4	5. D	6. D	7. 7.2×10^{-7}

8. $0 < x \le 6$	9. $\frac{32e^8}{3}$	10. A

11. $f(x) = x + \frac{3}{2}x^2 + \frac{3}{2}x^3 + \frac{9}{8}x^4 + \cdots + \frac{3^n x^{n+1}}{(n+1)n!}$	12. 6

Taylor Polynomial is a polynomial that will approximate other function's values in a region that is nearby the "center"
*a tangent line is essentially a first degree taylor polynomial.

n^{th} degree taylor polynomial:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Alternating Series Remainder:
Suppose an alternating series converges by AST. If the Series has Sum S, then $|R_n| = |S - S_n| \le |a_{n+1}|$

*This means that the **maximum error** for the n^{th} term partial Sum S_n is no greater than the absolute value of the first unused term a_{n+1}

Taylor Series: A General method for writing a power series representation for a function.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f^{(n)}$ represents the n^{th} derivative evaluated at f.

Power Series: Written in form $\sum_{n=0}^{\infty} a_n (x-c)^n$ where c and a_n (coefficients) are numbers:
*Taylor and Maclaurin series are special cases of power series
For a power series centered at c, precisely one of the following is true:
1) The series converges only at c (ALL power series converge at least at their center) (Radius of convergence = 0)
2) The series converges for all x (function and infinite series have exact same values everywhere) → Radius = ∞
3) The series converges within a certain Radius of Convergence such that series converges for $|x - c| < R$
→The **interval of Convergence** (I.O.C.) is
[(c − R , c + R)]
*Be sure to TEST convergence of endpoints
*Typically, you want to use the **RATIO TEST** to determine Radius of Convergence

Maclaurin Series : is the special case of Taylor series when **c = 0**.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Special Maclaurin Series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} \quad \text{IOC: All Reals}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} \quad \text{IOC: All Reals}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!} \quad \text{IOC: All Reals}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)} \quad \text{IOC}: -1 \le x \le 1$$

Geometric Series below based on

$$S = \frac{a_1}{1-r} \quad \text{IOC: } -1 < x < 1$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots (-1)^n x^n + \dots$$

IOC: $-1 < x < 1$

$$\frac{1}{x} = \frac{1}{1-[-(x-1)]} = 1 - (x-1) + (x-1)^2 - \dots (-1)^n (x-1)^n$$

IOC: $0 < x < 2$

$$\ln x = \int \frac{1}{x}\,dx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \dots \frac{(-1)^{n-1}(x-1)^n}{n}$$

IOC: $0 < x \le 2$

LaGrange Error Bound *This is similar to the Alternating Series Remainder. However, this method offers a way to determine the maximum error (remainder) when we do a Taylor polynomial approximation using a certain number of terms for a specific function.

$$R_n(x) = \left| \frac{|f^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} \right| \le \left| \frac{\max|f^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} \right|$$

* The remainder for an n^{th} degree polynomial is found by taking the $(n+1)^{st}$ (**first unused**) derivative at "z" *We are not expected to find the exact value of z. (If we could, then an approximation would not be necessary) *We want to maximize the $(n+1)^{st}$ derivative on the interval from [x , c] in order to find a safe upper bound for the $|f^{(n+1)}(z)|$ *The maximum error bound is the worst case scenario for the interval in which our actual approximation can live. **College Board will provide strictly increasing and decreasing functions. (So we only have to choose between f(c) and f(x) (the endpoints). This will allow us to determine the max value much more accurately.

Alternating Series Remainder:
Suppose an alternating series converges by AST
(such that $\lim_{n\to\infty} a_n = 0$ and a_n is decreasing), then

$$|R_n| = |S - S_n| \le |a_{n+1}|$$

*This means that the **maximum error** for the n^{th} term partial Sum S_n is no greater than the absolute value of the first unused term a_{n+1}