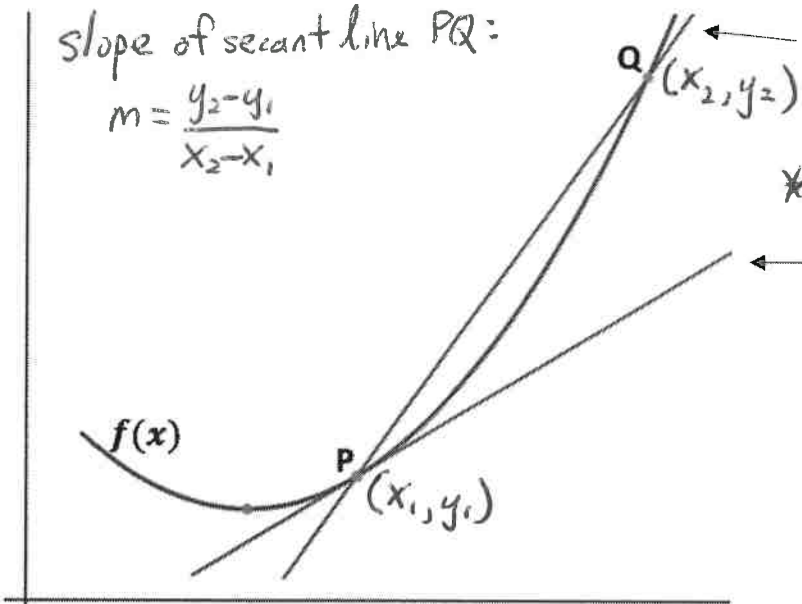
Key

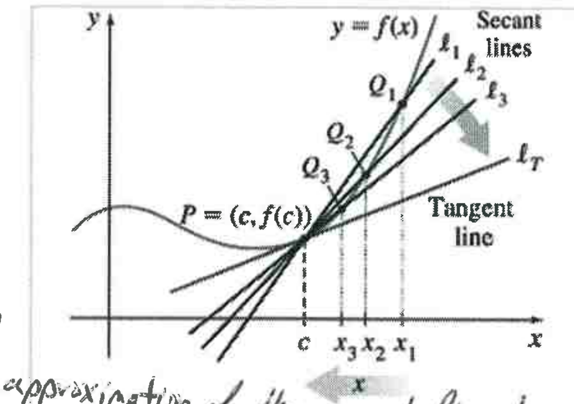
AP Calculus – 2.2 Notes - Limit Definition of a Derivative

Goal: To discover a formula to calculate the slope (steepness) of all tangent lines to a curved graph.

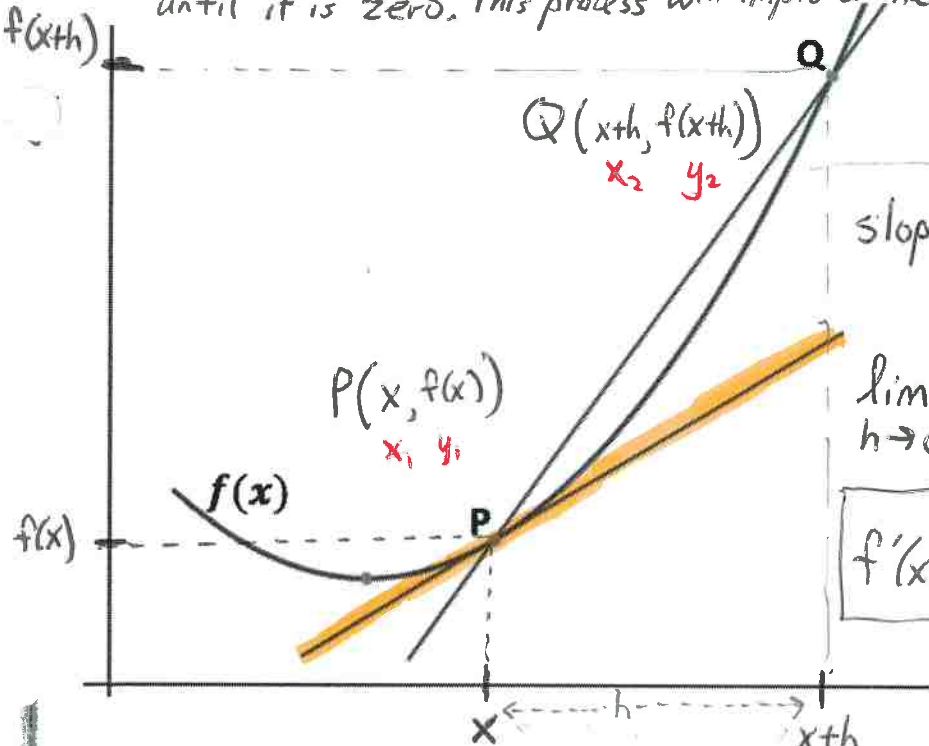


*The slope of this secant line is a rough approximation of the slope of this tangent line at point P.

*Improve the approximation of secant line PQ by choosing point Q closer to point P.



*Use limits to decrease the distance b/t P and Q until it is zero. This process will improve the approximation of the secant line slope until the value becomes the exact slope of the tangent line.



slope $\rightarrow \frac{y_2 - y_1}{x_2 - x_1} \rightarrow \frac{f(x+h) - f(x)}{x+h-x}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

A more robust version of the slope formula

General Limit Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Limit Definition of a derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

f '(x) is "f prime of x": This is the notation for the derivative function.

Derivative is the slope (steepness) of a curve at a single point

*The derivative function is a **slope-finding formula** for a curved graph, where the slope is of the curve is ever-changing.

General Limit Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Limit Definition of a derivative:

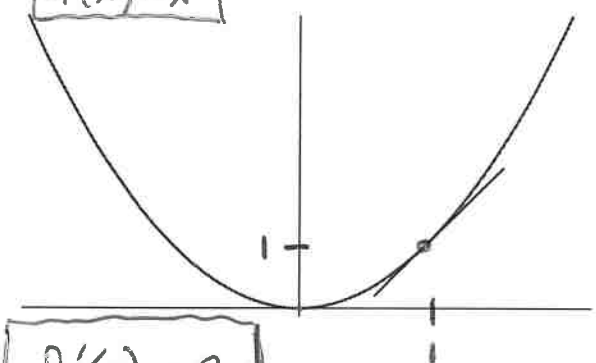
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example 1: (a) Find the general derivative of $f(x) = x^2$

(b) Write the equation of the tangent line to $f(x)$ at $x = 1$ (point-slope form: $\boldsymbol{y - y_1 = m(x - x_1)}$)

(c) Write the equation of the tangent line to $f(x)$ at $x = -5$

$\boxed{f(x) = x^2}$



$\boxed{f'(x) = 2x}$

a) $f(x) = x^2$

$f(\ ) = (\ )^2$

$f(x+h) = (x+h)^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \to \frac{0}{0}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)(x+h) - x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$f'(x) = \frac{2xh + h^2}{h} \to \lim_{h \to 0} \frac{\cancel{h}(2x+1)}{\cancel{h}} = 2x$$

b) $f(x) = x^2$

$f(1) = 1^2 = 1$

$f'(x) = 2x$

$f'(1) = 2(1) = 2$

point: $(1, 1)$

slope: $m = 2$

$y - y_1 = m(x - x_1)$

$\boxed{y - 1 = 2(x - 1)}$

c) Use $f(x) = x^2$ and $f'(x) = 2x$

$f(-5) = (-5)^2 = 25$

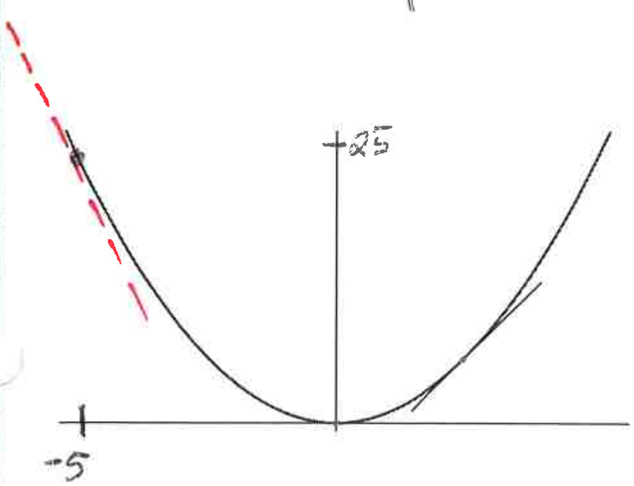
$f'(-5) = 2(-5) = -10$

point: $(-5, 25)$

slope: $m = -10$

$y - y_1 = m(x - x_1)$

$\boxed{y - 25 = -10(x + 5)}$



To Recap:

* $\boldsymbol{f(x)}$ is the **height-finding formula** (finds the y-value of graph at that point)

*Since $f(1) = 1$, this tells us that when x = 1, the height of the graph has a y-value of 1

***$\boldsymbol{f'(x)}$** is the **slope-finding formula** for the $f(x)$ graph.

*Since $f'(1) = 2$, this tells us that when x = 1, the slope of the tangent line to $f(x)$ has a slope (steepness) of 2.

General Limit Definition of the Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Alternate Limit Definition of a derivative:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example 2: (a) Find the general derivative of $f(x) = \sqrt{x}$

(b) Write the equation of the tangent line to $f(x)$ at $x = 2$ (point-slope form: $\boldsymbol{y - y_1 = m(x - x_1)}$)

a) $f(x) = \sqrt{x}$

$f(\ \ ) = \sqrt{(\ \ )}$

$f(x+h) = \sqrt{(x+h)}$

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \to \frac{0}{0}$

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$

$f'(x) = \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \to \lim_{h \to 0} \frac{\cancel{h}(1)}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$

$\lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \to \frac{1}{\sqrt{x} + \sqrt{x}} \to \frac{1}{2\sqrt{x}}$ $\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$

b) $f(x) = \sqrt{x}$

$f(2) = \sqrt{2}$

$f'(x) = 1/2\sqrt{x}$

$f'(2) = \frac{1}{2\sqrt{2}}$

point: $(2, \sqrt{2})$

slope: $m = \frac{1}{2\sqrt{2}}$

$y - y_1 = m(x - x_1)$

$\boxed{y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)}$

Example 3: Use the alternative derivative definition to find slope of $f(x) = \sqrt{x}$ at x = 2.

$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

*start by replacing $c = 2$ in formula

$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

$f(x) = \sqrt{x}$

$f(2) = \sqrt{2}$

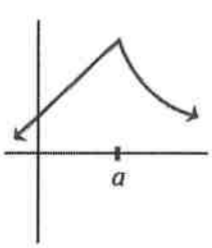
$f'(2) = \lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \to \frac{0}{0}$

$\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \cdot \frac{(\sqrt{x} + \sqrt{2})}{(\sqrt{x} + \sqrt{2})}$

$f'(2) = \lim_{x \to 2} \frac{\cancel{x-2}(1)}{\cancel{(x-2)}(\sqrt{x} + \sqrt{2})} \to \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$

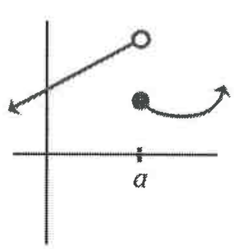
$\boxed{f'(2) = \frac{1}{2\sqrt{2}}}$

Differentiability: In order for a function to be **differentiable** (smooth curve) at a point a, then the graph must be continuous at that point, cannot contain a sharp turn & cannot have a vertical tangent at the point.



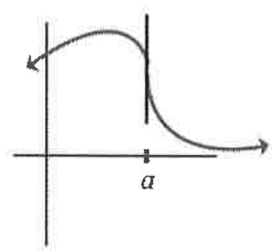
Cusp / Corner

$f'(a) = d.n.e.$ (does not exist)



Discontinuous

$f'(a) = dne$



Vertical Tangent

$f'(a) = dne$