AP Calculus – 2.4 Notes - Product Rule and Quotient Rule

Key

Product Rule

$$h(x) = f \cdot g$$

$$h'(x) = f' \cdot g + f \cdot g'$$

Find the derivative of each function.

1. $f(x) = 8x \sin x$

$f'(x) = 8 \cdot \sin x + 8x \cdot \cos x$

$\boxed{f'(x) = 8\sin x + 8x\cos x}$

2. $g(x) = 2e^x(\sqrt{x})$

$g(x) = 2e^x \cdot x^{1/2}$

$g'(x) = 2e^x \cdot x^{1/2} + 2e^x \cdot \frac{1}{2}x^{-1/2}$

$\boxed{g'(x) = 2e^x\sqrt{x} + \frac{e^x}{\sqrt{x}}}$

3. $h(x) = \left(\frac{1}{x}+1\right)(2x^2-5)$ $\rightarrow (x^{-1}+1)$

$h'(x) = -1x^{-2} \cdot (2x^2-5) + \left(\frac{1}{x}+1\right) \cdot 4x$

$h'(x) = \frac{2x^2-5}{-x^2} + 4 + 4x$

$h'(x) = \frac{2x^2}{-x^2} + \frac{5}{x^2} + 4 + 4x = -2 + \frac{5}{x^2} + 4 + 4x$

$\boxed{h'(x) = \frac{5}{x^2} + 4x + 2}$

The table below shows values of two differentiable functions f and g, as well as their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	−2	−1	2
−5	3	4	−2	5

4. $h(x) = 3f(x)g(x)$
Find $h'(2)$.

$h'(x) = 3f'(x) \cdot g(x) + 3f(x) \cdot g'(x)$

$h'(2) = 3f'(2)g(2) + 3f(2)g'(2)$

$h'(2) = 3(-2)(-1) + 3(4)(2) = \boxed{30}$

5. $r(x) = \left(\frac{f(x)}{2}+2\right)(3-g(x))$ $\rightarrow \left(\frac{1}{2}f(x)+2\right)(3-g(x))$
Find $r'(-5)$.

$r'(x) = \frac{1}{2}f'(x) \cdot (3-g(x)) + \left(\frac{1}{2}f(x)+2\right)(-g'(x))$

$r'(-5) = \frac{1}{2}f'(-5)(3-g(-5)) + \left(\frac{1}{2}f(-5)+2\right)(-g'(-5))$

$= \frac{1}{2}(4)(3--2) + \left(\frac{1}{2}(3)+2\right)(-5) = 2(5) + \frac{7}{2}(-5)$

$= \boxed{\frac{-15}{2}}$

Find the derivative of each function.

1. $f(x) = (2x-3)\sin x$

$f'(x) = 2 \cdot \sin x + (2x-3) \cdot \cos x$

$\boxed{f'(x) = 2\sin x + (2x-3)\cos x}$

2. $g(x) = 2x^3e^x$

$g'(x) = 6x^2 \cdot e^x + 2x^3 \cdot e^x$

$\boxed{g'(x) = 6x^2e^x + 2x^3e^x}$

3. $h(x) = 4\sqrt{x}\ln x$

$= (4x^{1/2})(\ln x)$

$h'(x) = 4 \cdot \frac{1}{2}x^{-1/2} \cdot \ln x + 4x^{1/2} \cdot \left(\frac{1}{x}\right)$

$h'(x) = \frac{2\ln x}{\sqrt{x}} + \frac{4}{\sqrt{x}} = \boxed{\frac{2\ln x + 4}{\sqrt{x}}}$

4. $f(x) = (4-5x)\cos x$

$f'(x) = (-5)\cos x + (4-5x)(-\sin x)$

$\boxed{f'(x) = -5\cos x - \sin x(4-5x)}$

5. $g(x) = 6\ln x \sin x$

$g'(x) = 6\left(\frac{1}{x}\right)\sin x + 6\ln x \cdot \cos x$

$\boxed{g'(x) = \frac{6\sin x}{x} + 6\ln x\cos x}$

6. $h(x) = 2e^x(x^2+x)$

$h'(x) = 2e^x(x^2+x) + 2e^x(2x+1)$

$h'(x) = 2e^x(x^2+x+2x+1)$

$\boxed{h'(x) = 2e^x(x^2+3x+1)}$

Quotient Rule

$$h(x)=\frac{f}{g}$$

$$h'(x)=\frac{f'g-fg'}{g^2}$$

Find the derivative of each function.

1. $y=\frac{\overbrace{2x^2}^{f}}{\underbrace{3x+1}_{g}}$

$$y'=\frac{\overbrace{4x}^{f'}\cdot\overbrace{(3x+1)}^{g}-\overbrace{2x^2}^{f}\cdot\overbrace{(3)}^{g'}}{\underbrace{(3x+1)^2}_{g^2}}\rightarrow\frac{12x^2+4x-6x^2}{(3x+1)^2}$$

$$\boxed{y'=\frac{6x^2+4x}{(3x+1)^2}}$$

2. $g(x)=\frac{\overbrace{3e^x}^{f}}{\underbrace{2x}_{g}}$

$$g'(x)=\frac{\overbrace{3e^x}^{f'}\cdot\overbrace{2x}^{g}-\overbrace{3e^x}^{f}\cdot\overbrace{2}^{g'}}{\underbrace{(2x)^2}_{g^2}}\rightarrow\frac{6xe^x-6e^x}{4x^2}$$

$$\boxed{g'(x)=\frac{6e^x(x-1)}{4x^2}}$$

3. $h(x)=\frac{\sin x}{2x^2-5}$

$$\boxed{h'(x)=\frac{\cos x(2x^2-5)-\sin x(4x)}{(2x^2-5)^2}}$$

4. $h(x)=\frac{3x+1}{2x^2}$

$$h'(x)=\frac{\overbrace{3}^{f'}\overbrace{(2x^2)}^{g}-\overbrace{(3x+1)}^{f}\overbrace{(4x)}^{g'}}{\underbrace{(2x^2)^2}_{g^2}}\rightarrow\frac{6x^2-12x^2-4x}{4x^4}$$

$$h'(x)=\frac{-6x^2-4x}{4x^4}$$

$$\boxed{h'(x)=\frac{-3}{2x^2}-\frac{1}{x^3}}$$

The table below shows values of two differentiable functions f and g, as well as their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	-1	2

5. $h(x)=\frac{f(x)}{3g(x)}$
Find $h'(2)$.

$$h'(x)=\frac{f'(x)\cdot 3g(x)-f(x)\cdot 3g'(x)}{[3g(x)]^2}$$

$$h'(2)=\frac{f'(2)\cdot 3g(2)-f(2)\cdot 3g'(2)}{9[g(2)]^2}=\frac{(-2)(3)(-1)-4(3)(2)}{9(-1)^2}$$

$$=\frac{6-24}{9}\rightarrow\frac{-18}{9}\qquad\boxed{h'(2)=-2}$$

6. $r(x)=-\frac{g(x)}{1-f(x)}\rightarrow\frac{-g(x)}{1-f(x)}$
Find $r'(2)$.

$$r'(x)=\frac{\overbrace{-g'(x)}^{f'}\overbrace{(1-f(x))}^{g}-\overbrace{-g(x)}^{f}\cdot\overbrace{(-f'(x))}^{g'}}{\underbrace{(1-f(x))^2}_{g^2}}$$

$$r'(2)=\frac{-g'(2)[1-f(2)]-g(2)\cdot f'(2)}{[1-f(2)]^2}$$

$$r'(2)=\frac{-2(1-4)-(-1)(-2)}{(1-4)^2}$$

$$=\frac{6-2}{9}=\boxed{\frac{4}{9}}$$

$$\boxed{r'(2)=4/9}$$

Find the derivative of each function.

1. $h(x)=\frac{4x-1}{3x+2}$

$$h'(x)=\frac{4(3x+2)-(4x-1)(3)}{(3x+2)^2}$$

$$h'(x)=\frac{12x+8-12x+3}{(3x+2)^2}\rightarrow\boxed{\frac{11}{(3x+2)^2}}$$

2. $g(x)=\frac{\sin x}{x}$

$$g'(x)=\frac{(\cos x)(x)-(\sin x)(1)}{x^2}$$

$$\boxed{g'(x)=\frac{x\cos x-\sin x}{x^2}}$$

3. $h(x)=\frac{x^3+2x^2-x}{2x}$

$$\cancel{h'(x)=\frac{(3x^2+4x-1)(2x)-(x^3+2x^2-x)(2)}{(2x)^2}}$$

simplify first!

$$h(x)=\frac{1}{2}x^2+x-\frac{1}{2}\qquad\boxed{h'(x)=x+1}$$

Product and Quotient Rule Practice Problems:

13.

x	$d(x)$	$d'(x)$	$h(x)$	$h'(x)$
1	-3	-2	4	3

a. $a(x) = d(x)h(x)$
Find $a'(1)$.

$a'(t) = d'(x)h(x) + d(x)\cdot h'(x)$

$a'(1) = d'(1)h(1) + d(1)h'(1)$

$a'(1) = (-2)(4) + (-3)(3) = \boxed{-17}$

$-8 \quad - 9$

b. $b(x) = -\overbrace{d(x)}^{f}\overbrace{h(x)}^{g}$
Find $b'(1)$.

$b'(x) = -\overbrace{d'(x)}^{f'}\cdot \overbrace{h(x)}^{g} + -\overbrace{d(x)}^{f}\cdot \overbrace{h'(x)}^{g'}$

$b'(1) = -d'(1)h(1) - d(1)h'(1)$

$b'(1) = -(-2)(4) - (-3)(3)$

$= 8 + 9 = \boxed{17}$

c. $c(x) = \left(2 - \frac{d(x)}{2}\right)(6 - h(x))$
Find $c'(1)$.

$\left(2 - \frac{1}{2}d(x)\right)\left(6 - h(x)\right)$

$c'(x) = -\frac{1}{2}d'(x)(6 - h(x)) + \left(2 - \frac{1}{2}d(x)\right)(-h'(x))$

$c'(1) = -\frac{1}{2}d'(1)(6 - h(1)) + \left(2 - \frac{1}{2}d(1)\right)(-h'(1))$

$= (1)(6-4) + (2 + 3/2)(-3)$

$= \boxed{-17/2}$

Find the equation of the tangent line at the given x-value.

14. $f(x) = 8\sin x \cos x$ at $x = \frac{\pi}{3}$

$f'(x) = 8\cos x(\cos x) + 8\sin x(-\sin x)$

$f'(x) = 8\cos^2 x - 8\sin^2 x$

$f'(\pi/3) = 8(\cos \pi/3)^2 - 8(\sin \pi/3)^2$

$= 8\left(\frac{1}{2}\right)^2 - 8\left(\frac{\sqrt{3}}{2}\right)^2$

$8\left(\frac{1}{4}\right) - 8\left(\frac{3}{4}\right) = -4$

$f(\pi/3) =$

$8\sin \pi/3 \cos \pi/3$

$= 8\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$= 2\sqrt{3}$

point: $(\pi/3, 2\sqrt{3})$

slope: $m = -4$

$\rightarrow \boxed{y - 2\sqrt{3} = -4(x - \pi/3)}$

14b) $g(x) = -2xe^x$ at $x = 0$

$g'(x) = -2e^x + -2xe^x$

$g'(0) = -2e^0 - 2(0)e^0$

$g'(0) = -2$

$g(0) = -2(0)e^0 = 0$

point: $(0, 0)$

slope: $m = -2$

$y - 0 = -2(x - 0)$

$\boxed{y = -2x}$

15. What is the instantaneous rate of change at $x = 4$ of the function $f(x) = \frac{x^2-1}{x-2}$? (Quotient Rule!)

$f'(x) = \frac{2x(x-2) - (x^2-1)(1)}{(x-2)^2} \rightarrow \frac{2x^2 - 4x - x^2 + 1}{(x-2)^2} \rightarrow f'(4) = \frac{2(4)^2 - 4(4) - (4)^2 + 1}{(4-2)^2} \rightarrow \boxed{\frac{1}{4}}$

(A) $-\frac{15}{2}$ $\quad$ (B) $\frac{1}{4}$ $\quad$ (C) $\frac{1}{2}$ $\quad$ (D) $\frac{15}{2}$

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16. Let f and g be differentiable functions with the following properties:

I. $f(x) < 0$ for all x

II. $g(5) = 2$

If $h(x) = \frac{f(x)}{g(x)}$ and $h'(x) = \frac{f'(x)}{g(x)}$, then $g(x) =$

* If $g'(x) = 0$, then $g(x)$ is a constant. Since $g(5) = 2$, then $\boxed{g(x) = 2}$

$h'(x) = \frac{f'(x)g(x) - f(x)\boxed{g'(x)}}{[g(x)]^2} \rightarrow \frac{f'(x)g(x) - f(x)\boxed{\cdot 0}}{[g(x)]^2} \rightarrow \frac{f'(x)}{g(x)}$

(A) $\frac{1}{f'(x)}$ $\quad$ (B) $f(x)$ $\quad$ (C) $-f(x)$ $\quad$ (D) 0 $\quad$ (E) 2

17. The function f is defined by $f(x) = \frac{x}{x+4}$. What points (x, y) on the graph of f have the property that the line tangent to f at (x, y) has a $\boxed{\text{slope of } \frac{1}{9}}$?

(A) $\left(2, \frac{1}{3}\right)$ only

(B) $\left(\frac{1}{9}, \frac{1}{13}\right)$ only

(C) $\left(2, \frac{1}{3}\right)$ and $\left(-10, \frac{5}{3}\right)$

(D) $\left(2, \frac{1}{3}\right)$ and $(-2, -1)$

(E) There are no such points.

* quotient Rule

$$f'(x) = \frac{(1)(x+4) - x(1)}{(x+4)^2}$$

$$f'(x) = \frac{x+4-x}{(x+4)^2} \rightarrow \frac{4}{(x+4)^2}$$

$$\boxed{f'(x)} = \frac{4}{(x+4)^2}$$

$$\frac{1}{9} = \frac{4}{(x+4)^2}$$

$$(x+4)^2 = 36$$

$$\sqrt{(x+4)^2} = \sqrt{36}$$

$$x+4 = \pm 6$$

$$x = -4+6 \rightarrow 2$$

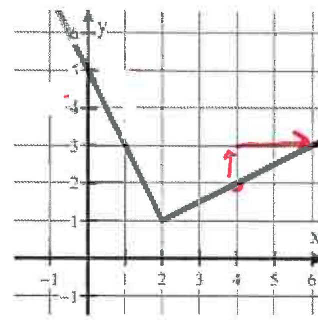
$$x = -4-6 \rightarrow -10$$

points:

$\left(2, \frac{1}{3}\right)$ and

$\left(-10, \frac{5}{3}\right)$

18. The graph of a function f is shown to the right. Let $g(x) = \frac{\overbrace{x^2-1}^{f}}{\underbrace{f(x)}_{g}}$. What is the value of $g'(4)$? *quotient Rule



Graph of f

$f'(4) = \frac{1}{2}$

$$g'(x) = \frac{\overbrace{(2x)}^{f'}\overbrace{(f(x))}^{g} - \overbrace{(x^2-1)}^{f} \cdot \overbrace{f'(x)}^{g'}}{\underbrace{[f(x)]^2}_{g^2}}$$

$$g'(4) = \frac{2(4)f(4) - [4^2-1] \cdot f'(4)}{[f(4)]^2}$$

$$g'(4) = \frac{8(2) - (15)\left(\frac{1}{2}\right)}{(2)^2}$$

$$g'(4) = \frac{16 - \frac{15}{2}}{4}$$

$$g'(4) = \frac{\frac{17}{2}}{4} \rightarrow \frac{17}{2} \cdot \frac{1}{4} = \frac{17}{8}$$

$$\boxed{g'(4) = \frac{17}{8}}$$

19. The graphs of f and g are shown to the right. If $h(x) = 4\overbrace{f(x)}^{f}\overbrace{g(x)}^{g}$, then $h'(1) =$

*product Rule

(A) -22

(B) -4

(C) 0

(D) 4

$\boxed{\text{(E) } 46}$

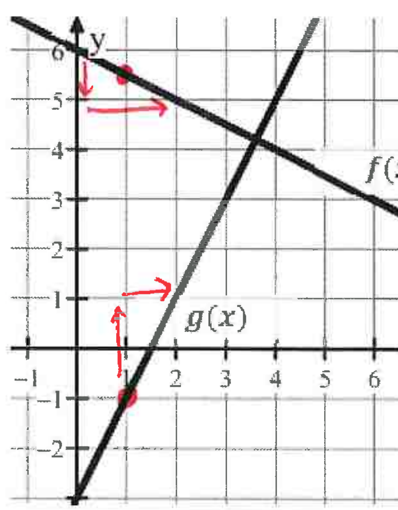
$$h'(x) = \overbrace{4f'(x)}^{f'} \cdot \overbrace{g(x)}^{g} + \overbrace{4f(x)}^{f} \cdot \overbrace{g'(x)}^{g'}$$

$$h'(1) = 4f'(1)g(1) + 4f(1)g'(1)$$

$$h'(1) = 4\left(\frac{-1}{2}\right)(-1) + 4(5.5)(2)$$

$$h'(1) = 2 + 44$$

$$\boxed{h'(1) = 46}$$



$f'(1) = \frac{-1}{2}$

$g'(1) = \frac{2}{1} = 2$