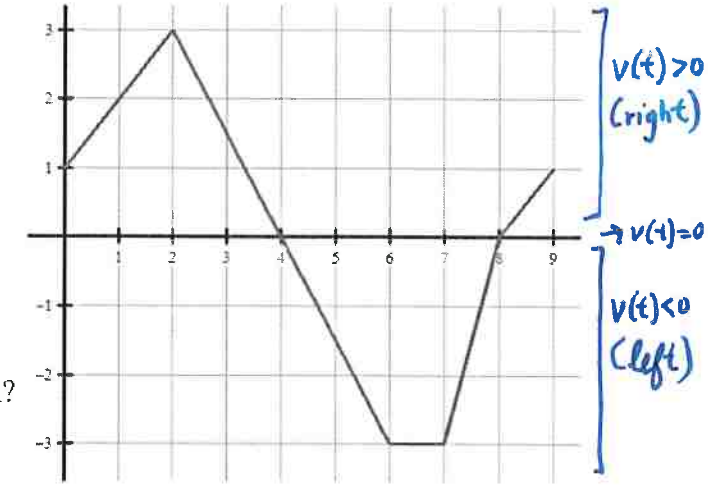
BC Calculus 4.1-4.4 Quiz Review (Calculators Allowed)

Key

(Related Rates, Particle Motion, Linear Approximation, L'Hopital's Rule)

1. The figure shows the velocity $v = \frac{ds}{dt} = f(t)$ of a body moving along a coordinate line in meters per second.



a) When does the body reverse direction?

$t=4$, $t=8$ b/c $v(t)$ change signs

b) When is the body moving at a constant speed?

$(6,7)$

c) What is the body's maximum speed?

3 meters/sec

d) At what time interval(s) is the body slowing down?

* $v(t)$ and $a(t)$ have opposite signs

from $(2,4)$ and $(7,8)$

Find the following. Use L'Hospital's when possible.

2. $\lim\limits_{x\to 2} \frac{x-2}{x^2-7x+10} \to \frac{0}{0}$

$\lim\limits_{x\to 2} \frac{1}{2x-7} \to \boxed{\frac{1}{-3}}$

3. $\lim\limits_{x\to 0} \frac{3x^2}{e^x-1-x} \to \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{6x}{e^x-1} \to \frac{0}{1-1} \to \frac{0}{0}$

$\lim\limits_{x\to 0} \frac{6}{e^x} \to \frac{6}{e^0} = \boxed{6}$

4. $\frac{d}{dx} \frac{3x-2}{5x+1}$

* quotient Rule

$\frac{f'g-fg'}{g^2}$

$\frac{(3)(5x+1)-(3x-2)(5)}{(5x+1)^2}$

$\frac{15x+3-15x+10}{(5x+1)^2} \to \boxed{\frac{13}{(5x+1)^2}}$

5. If the length l of a rectangle is decreasing at a rate of 2 inches per minute while its width w is increasing at a rate of 2 inches per minute, which of the following must be true about the area A of the rectangle?

(A) A is always increasing.

(B) A is always decreasing.

(C) A is increasing only when $l > w$.

(D) A is increasing only when $l < w$.

(E) A remains constant.

$A = \underbrace{(l)}_{f}\underbrace{(w)}_{g}$ ← * product Rule

$\frac{dl}{dt} = -2 \quad \frac{dw}{dt} = 2$

$\frac{dA}{dt} = \overbrace{\frac{dl}{dt}}^{f'} \cdot \overbrace{w}^{g} + \overbrace{l}^{f} \cdot \overbrace{\frac{dw}{dt}}^{g'}$

$\frac{dA}{dt} = -2w + l(2)$

$= 2l - 2w$

$\boxed{\frac{dA}{dt} = 2(l-w)}$

6) A particle moves along the x-axis so that at times $t \geq 0$, its position is given by

$x(t) = t^3 - 3t^2 - 9t + 2$ (in meters)

a) Find the velocity and acceleration function

$\boxed{v(t) = 3t^2 - 6t - 9}$ $\boxed{a(t) = 6t - 6}$

b) What is its velocity at $t = 2$ seconds?

$v(2) = 3(2)^2 - 6(2) - 9 = 12 - 12 - 9$

$\boxed{v(2) = -9 \text{ m/s}}$

c) What is its acceleration at t = 4 seconds?

$a(4) = 6(4) - 6$

$\boxed{a(4) = 18 \text{ m/s}^2}$

d) At what times does the particle change directions? Justify

*set $v(t) = 0$ to find first particle at rest

$0 = 3(t^2 - 2t - 3)$

$0 = 3(t-3)(t+1)$

$t = 3, -1$

$v(t)$: − on (0, 3), + after 3; 0, 3, 1, 4

$\boxed{\text{change direction at } t = 3 \text{ since } v(t) \text{ change signs}}$

e) At t = 0, is the particle moving to the right or to the left? Justify.

$v(0) = 3(0)^2 - 6(0) - 9$

Since $v(0) < 0$, particle is moving left at $t = 0$

f) Find the average velocity of particle in [1, 3]

Avg. velocity $= \dfrac{\text{change in position}}{\text{change in time}} \rightarrow \dfrac{x(3) - x(1)}{3 - 1}$

$x(3) = -25$

$x(1) = -9$

Avg. velocity $= \dfrac{-25 - (-9)}{3 - 1} = \boxed{-8 \text{ m/s}}$

g) What is displacement of particle from t = 1 to t = 4? Show work.

*displacement = final position − initial position

$= x(4) - x(1)$

$= -18 - (-9)$

$= \boxed{-9 \text{ meters}}$

h) What is the total distance of particle from t = 1 to t = 4? Show work.

*we include t=3 because there is a change of direction at t=3.

$v(t)$: − on (0, 3), + after 3

$x(1) = -9$
$x(3) = -25$ > 16
$x(4) = -18$ > 7

$\boxed{\text{Total distance is 23 meters}}$

i) Is velocity increasing or decreasing at t = 2? Justify.

*Just look at the sign of acceleration.

$a(2) = 6 \text{ m/s}^2$. Therefore velocity is increasing since $a(t) > 0$

j) Is the speed increasing or decreasing at t = 4? Justify

*compare signs of $v(t)$ and $a(t)$.

$v(4) = 15$

$a(4) = 18$

$\boxed{\text{speed is increasing at } t = 4 \text{ since } v(t) \text{ and } a(t) \text{ have same signs at } t = 4}$

k) Avg. acceleration: $\dfrac{v(2) - v(1)}{2 - 1} \rightarrow \dfrac{-9 - (-12)}{1} = \dfrac{3}{1} = \boxed{3 \text{ m/s}^2}$

7) The function $f(x) = (1-\sin x)^2$ is concave up at $x = \frac{\pi}{6}$?

a. What is the estimate for $f(0.5)$ using the local linear approximation for f at $x = \frac{\pi}{6}$?

$f(\frac{\pi}{6}) = [1-\sin(\frac{\pi}{6})]^2 = [1-\frac{1}{2}]^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$f'(x) = 2[1-\sin x]\cdot(-\cos x)$

$f'(\pi/6) = 2(1-\sin(\pi/6))(-\cos(\pi/6))$

$f'(\pi/6) = 2(1-\frac{1}{2})(-\frac{\sqrt{3}}{2}) = 2(\frac{1}{2})(\frac{-\sqrt{3}}{2}) = \frac{-\sqrt{3}}{2}$

point: $(\pi/6, 1/4)$ slope: $m = \frac{-\sqrt{3}}{2}$

$y - \frac{1}{4} = \frac{-\sqrt{3}}{2}(x - \pi/6)$

$y = \frac{-\sqrt{3}}{2}(x-\pi/6) + \frac{1}{4}$

$f(0.5) \approx y(0.5) = \frac{-\sqrt{3}}{2}(0.5 - \frac{\pi}{6}) + \frac{1}{4}$

$= \boxed{0.270}$

b. Is it an underestimate or overestimate? Explain.

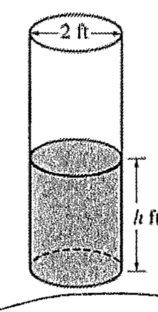
Underestimate since $f(x)$ is concave up.

8) A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.

a) Find $\frac{dV}{dt} = \_\_?$ $h = 4$ ft. radius $= 1$ ft (constant)



$V = \pi r^2 h$

$V = \pi (1)^2 h$

$V = \pi h$

$\frac{dV}{dt} = \pi\left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \pi \frac{-1}{10}\sqrt{h}$

$\frac{dV}{dt} = \pi \cdot \frac{-1}{10}\sqrt{4}$

$\frac{dV}{dt} = \frac{-\sqrt{4}\pi}{10}$

$\boxed{\frac{dV}{dt} = -\frac{\pi}{5} \text{ ft}^3/\text{sec}}$

b) * rate of change increasing/decreasing refers to $\frac{d^2h}{dt^2}$ (at $h=3$)

$\frac{dh}{dt} = \frac{-1}{10}\sqrt{h} = \frac{-1}{10}(h)^{1/2}$

$\frac{d^2h}{dt^2} = \frac{-1}{10}\cdot\frac{1}{2}h^{-1/2}\left(\frac{dh}{dt}\right)$

$\frac{d^2h}{dt^2} = \frac{-1}{20}\left(\frac{1}{\sqrt{h}}\right)\left(\frac{-1}{10}\sqrt{h}\right)$

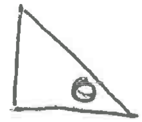
$\frac{d^2h}{dt^2} = \frac{1}{200} > 0$

Rate of change of height of water is increasing since $\frac{d^2h}{dt^2} > 0$

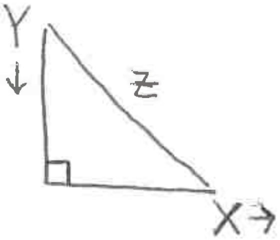
9) Person X and Person Y are walking on straight streets that meet at right angles. Y travels south and approaches the intersection at 2m/s. Person X travels east and moves away from the intersection at 1m/s.

a) Find the rate at which the distance (Z) between Person X and Y is changing when Y is 10m from the intersection and X is 20 meters from the intersection.

b) At what rate is the angle θ changing at the same moment?



c) Find the rate of change of Area triangle at the same moment.



$x = 20 \quad dx/dt = 1$

$y = 10 \quad dy/dt = -2$

$z = 10\sqrt{5} \quad dz/dt = \_\_\_$

$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

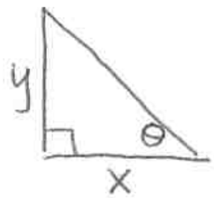
$2(20)(1) + 2(10)(-2) = 2(10\sqrt{5})\left(\frac{dz}{dt}\right)$

$40 - 40 = 20\sqrt{5}\left(\frac{dz}{dt}\right)$

$0 = 20\sqrt{5}\left(\frac{dz}{dt}\right)$

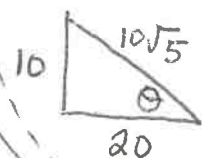
a) $\boxed{\frac{dz}{dt} = 0 \text{ m/s}}$

b)



$\tan\theta = \frac{y}{x}$

$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{\left(\frac{dy}{dt}\right)(x) - y\left(\frac{dx}{dt}\right)}{x^2}$



$\sec\theta = \frac{10\sqrt{5}}{20}$

$\sec\theta = \frac{\sqrt{5}}{2}$

$\left(\frac{\sqrt{5}}{2}\right)^2 \frac{d\theta}{dt} = \frac{(-2)(20) - (10)(1)}{(20)^2}$

$\left(\frac{5}{4}\right)\frac{d\theta}{dt} = \frac{-50}{400}$

$\frac{d\theta}{dt} = \frac{-50}{400} \cdot \frac{4}{5}$

$\boxed{\frac{d\theta}{dt} = \frac{-10}{100} = \frac{-1}{10} \text{ rad/sec}}$

c) $A = \frac{1}{2}xy$ (f = $\frac{1}{2}x$, g = y)

$\frac{dA}{dt} = \underbrace{\frac{1}{2}\left(\frac{dx}{dt}\right)}_{f'}\underbrace{(y)}_{g} + \underbrace{\frac{1}{2}x}_{f} \cdot \underbrace{\frac{dy}{dt}}_{g'}$

$\frac{dA}{dt} = \frac{1}{2}(1)(10) + \frac{1}{2}(20)(-2)$

$\frac{dA}{dt} = 5 - 20 = -15$

$\boxed{\frac{dA}{dt} = -15 \text{ m}^2/\text{sec}}$

Types of Related Rates Problems to Review:

1) Pythagorean Theorem Problems

2) Area of (Right) Triangle

3) Sphere (Volume & Surface Area) – formulas will be provided

4) Trig Related Rates

5) Cone Problems (formula will be provided)

6) Cylinder Problems (formula will be provided)

7) Similar Right Triangle (shadow problem)

*Formulas to memorize: Pythagorean theorem, Area and circumference of circle, area of square, area of triangle, SOH-CAH-TOA