BC Calculus Ch. 8.6 Notes   Integration by Tables and Other Integration Techniques

Integration by Tables

Unfortunately, we are still very limited in the number of total functions that we can integrate. ome functions are so complex, that there really isn't an effective procedure to integrate them. uckily, over the course of history, some very insightful mathematicians discovered patterns in the way certain functions can be integrated and developed a "formula" for how a certain function can be integrated.

In Appendix B of your textbook, you will see these formulas. The formulas in Appendix B are classified according to the forms involving the following:

u^n	$(a+bu)$
$(a+bu+cu^2)$	$\sqrt{a+bu}$
$(a^2 \pm u^2)$	$\sqrt{u^2 \pm a^2}$
$\sqrt{a^2-u^2}$	Trigonometric functions
Inverse trigonomeric functions	Exponential functions
Logarithmic functions	

Example 1: Integration by Tables

Find $\int \frac{dx}{x\sqrt{x-1}}$

Example 2: Integration by Tables

Find $\int x\sqrt{x^4-9}\,dx$

Example 3: Integration by Tables

Find $\int \frac{x}{1+e^{-x^2}}\,dx$

Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x)\,dx = g(x) + \int h(x)\,dx$. Such integration formulas are called **reduction formulas** because they reduce a given integral to the sum of a function and a simpler integral.

Ex. 4: Using a Reduction Formula.

Find $\int x^3 \sin x\,dx$

Ex.5: Using a Reduction Formula.

Find $\int \frac{\sqrt{3-5x}}{2x}\,dx$

Rational Functions of Sine and Cosine

Ex. 6: Integration by Tables.

Find $\int \frac{\sin 2x}{2+\cos x}\,dx$

SUBSTITUTION FOR RATIONAL FUNCTIONS OF SINE AND COSINE

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1+\cos x} = \tan\frac{x}{2}$$

yields

$$\cos x = \frac{1-u^2}{1+u^2},\quad \sin x = \frac{2u}{1+u^2},\quad \text{and}\quad dx = \frac{2\,du}{1+u^2}$$

Key

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Integration by Tables

Unfortunately, we are still very limited in the number of total functions that we can integrate. Some functions are so complex, that there really isn't an effective procedure to integrate them. Luckily, over the course of history, some very insightful mathematicians discovered patterns in the way certain functions can be integrated and developed a "formula" for how a certain function can be integrated.

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Example 1: Integration by Tables

Find $\int \frac{dx}{x\sqrt{x-1}}$ * involves form $\sqrt{a+bu}$

Formula #17 $(a<0)$ $\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C$

$a=-1$, $b=1$, $u=x$ $du=dx$

$= \frac{2}{\sqrt{1}} \arctan \sqrt{\frac{-1+1x}{-(-1)}} + C$

$= \boxed{2\arctan\sqrt{x-1} + c}$

Example 2: Integration by Tables

Find $\int x\sqrt{x^4-9}\,dx$

* involves $\sqrt{u^2 \pm a^2}$, $a>0$

* Formula #26

$\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left[u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u+\sqrt{u^2 \pm a^2}\right|\right] + C$

$u=x^2$

$du=2x\,dx$

$\frac{du}{2x}=dx$

$a=3$

$\int x\sqrt{u^2-a^2}\cdot\frac{du}{2x} = \frac{1}{2}\int\sqrt{u^2-a^2}\,du$

$= \frac{1}{2}\cdot\frac{1}{2}\left[x^2\sqrt{x^4-9} - 3^2\ln\left|x^2+\sqrt{x^4-9}\right|\right] + C$

$= \boxed{\frac{1}{4}\left[x^2\sqrt{x^4-9} - 9\ln\left|x^2+\sqrt{x^4-9}\right|\right) + C}$

Example 3: Integration by Tables

Find $\int \frac{x}{1+e^{-x^2}}\,dx$ * involves e^u

Formula #84 $\int \frac{1}{1+e^u}\,du = u - \ln(1+e^u) + C$

$u=-x^2$

$\frac{du}{dx}=-2x$

$dx=\frac{du}{-2x}$

$\int \frac{\cancel{x}}{1+e^u}\cdot\frac{du}{-2\cancel{x}} = \frac{-1}{2}\int\frac{du}{1+e^u}$ $\longrightarrow$

$-\frac{1}{2}\left[-x^2 - \ln\left(1+e^{-x^2}\right)\right] + C$

$\boxed{\frac{1}{2}\left[x^2 + \ln\left(1+e^{-x^2}\right)\right] + C}$

Reduction Formulas

Several of the integrals in the integration tables have the form $\int f(x)\,dx = g(x) + \int h(x)\,dx$. Such integration formulas are called **reduction formulas** because they reduce a given integral to the sum of a function and a simpler integral.

Ex. 4: Using a Reduction Formula.

* a) Formula #54

Find $\int x^3 \sin x\,dx$ $\quad \int u^n \sin u\,du = -u^n \cos u + n\int u^{n-1}\cos u\,du$

$u = x \quad du = dx, \; n = 3$

$= -x^3\cos x + 3\underbrace{\int x^2 \cos x\,dx}$

b) Formula #55 $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1}\sin u\,du$

$u = x, \; n = 2$

$3\left[x^2 \sin x - 2\underbrace{\int x^1 \sin x\,dx}\right]$

c) Formula #52 $\int u\sin u\,du = \sin u - u\cos u + C$

$= 2\left[\sin x - x\cos x\right]$

$= \boxed{-x^3\cos x + 3x^2\sin x - 6\sin x + 6x\cos x + C}$

Ex.5: Using a Reduction Formula.

Find $\int \frac{\sqrt{3-5x}}{2x}\,dx$ $\quad a = 3, \; b = -5$

$u = x \quad du = dx$

* Formula #19

$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{1}{u\sqrt{a+bu}}\,du$

$\frac{1}{2}\int \frac{\sqrt{3-5x}}{x}\,dx = \frac{1}{2}\left[2\sqrt{3-5x} + 3\underbrace{\int \frac{dx}{x\sqrt{3-5x}}}\right]$

* Formula #17, $a > 0$ $\quad a = 3, \; b = -5, \; u = x$

$\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}}\right| + C$

$= \boxed{\sqrt{3-5x} + \frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3-5x} - \sqrt{3}}{\sqrt{3-5x} + \sqrt{3}}\right| + C}$

Rational Functions of Sine and Cosine

Ex. 6: Integration by Tables.

Find $\int \frac{\sin 2x}{2+\cos x}\,dx$ $= \int \frac{2\sin x \cdot \cos x}{2+\cos x}\,dx$

* Formula #3:

$\int \frac{u\,du}{a+bu} = \frac{1}{b^2}\left(bu - a\ln|a+bu|\right) + C$

$a = 2, \; b = 1, \; u = \cos x \quad \frac{du}{dx} = -\sin x, \; dx = \frac{du}{-\sin x}$

$\int \frac{2\cancel{\sin x}\cdot u}{2+u}\cdot\frac{du}{-\cancel{\sin x}} = -2\int \frac{u}{2+u}\,du = -2\cdot\frac{1}{1^2}\left(1\cos x - 2\ln|2 + 1\cos x|\right) + C$

$= \boxed{-2\cos x + 4\ln|2+\cos x| + C}$

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