2017

9.10 Taylor and Maclaurin Series p.673 #1-40

Find Taylor series centered at c.

2) $f(x) = e^{-4x}$ $c = 0$ * Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-4x} = \sum_{n=0}^{\infty} \frac{(-4x)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (4x)^n}{n!}}$$

4) $f(x) = \sin x$ $c = \pi/4$

$f(x) = \sin x$ $f(\pi/4) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $f'(\pi/4) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $f''(\pi/4) = \frac{-\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $f'''(\pi/4) = -\frac{\sqrt{2}}{2}$

$f^4(x) = \sin x$ $f^4(\pi/4) = \frac{\sqrt{2}}{2}$

$$\sin x = \sum_{n=0}^{\infty} \frac{f^n(\pi/4)\left[x - \pi/4\right]^n}{n!}$$

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi/4) - \frac{\frac{\sqrt{2}}{2}}{2!}(x - \pi/4)^2$$

$$= \frac{\sqrt{2}}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^{\frac{n(n-1)}{2}}\left[x - \pi/4\right]^{n+1}}{(n+1)!} + 1\right]$$

6) $f(x) = \frac{1}{1-x}$ $c = 2$

$f(x) = (1-x)^{-1}$ $f(2) = -1$

$f'(x) = -1(1-x)^{-2}(-1) = 1(1-x)^{-2}$ $f'(2) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$ $f''(2) = -2$

$f'''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$ $f'''(2) = 6$

$$\sum_{n=0}^{\infty} \frac{f^n(2)(x-2)^n}{n!}$$

$$= -1 + 1(x-2) + \frac{-2}{2!}(x-2)^2 + \frac{6}{3!}(x-2)^3$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n}$$

8) $f(x) = e^x$ $\quad c=1$

$f(1)=e$

$f'(x)=e^x \quad f'(1)=e$

$f''(x)=e^x \quad f''(1)=e$

$f'''(x)=e^x \quad f'''(1)=e$

$$\sum_{n=0}^{\infty} \frac{f^n(1)(x-1)^n}{n!}$$

$$= e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{e(x-1)^n}{n!}}$$

10) $f(x) = \ln(x^2+1)$ $\quad c=0 \quad f(0)=0$

$f'(x) = \frac{2x}{x^2+1} \quad f'(0)=0$

$f''(x) = \frac{2-2x^2}{(x^2+1)^2} \quad f''(0)=2$

$f'''(x) = \frac{4x(x^2-3)}{(x^2+1)^3} \quad f'''(0)=0$

$f^4(x) = \frac{12(-x^4+6x^2-1)}{(x^2+1)^4} \quad f^4(0)=-12$

$f^5(x) = \frac{48x(x^4-10x^2+5)}{(x^2+1)^5} \quad f^5(0)=0$

$f^6(x) = \frac{-240(5x^6-15x^4+15x^2-1)}{(x^2+1)^6} \quad f^6(0)=240$

$$\sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

$$= 0 + 0x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 - \frac{12}{4!}x^4 + \frac{0}{5!}x^5 + \frac{240}{6!}x^6$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}}$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots$$

12) $f(x) = \tan x$ $\quad c=0 \quad f(0)=0$ Maclaurin series

$f'(x) = \sec^2 x \quad f'(0)=1$

$f''(x) = 2\sec^2 x \tan x \quad f''(0)=0$

$f'''(x) = 2[\sec^4 x + 2\sec^2 x \tan^2 x] \quad f'''(0)=2$

$f^4(x) = 8[\sec^4 x \tan x + \sec^2 x \tan^3 x] \quad f^4(0)=0$

$f^5(x) = 8[2\sec^6 x + 11\sec^4 x \tan^2 x + 2\sec^2 x \tan^4 x] \quad f^5(0)=16$

$$\tan x = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{n!}$$

$$= 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5$$

$$= \boxed{x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots}$$

14) $f(x)=e^{-2x}$ $\quad \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$ $\quad f^{n+1}(x)=(-2)^{n+1}e^{-2x}$

$$R_n(x)=\frac{f^{n+1}(z)}{(n+1)!}(x)^{n+1}=\left|\frac{-2^{n+1}e^{-2z}}{(n+1)!}(x)^{n+1}\right|$$

$$\lim_{n\to\infty}\frac{-2^{n+1}x^{n+1}}{(n+1)!}=0, \text{ so } R_n(x)=0 \text{ as } n\to\infty$$

16) $f(x)=\cosh x \rightarrow \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ (Inverse hyperbolic cosine function)

$$|R_n(x)|=\left|\frac{f^{n+1}(z)}{(n+1)!}x^{n+1}\right|=\left|\frac{\sinh(z)}{(n+1)!}x^{n+1}\right|\to 0 \text{ as } n\to\infty$$

Binomial Theorem $\quad (a+b)^n=\binom{n}{0}a^1+\binom{n}{1}a^{n-1}b+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}b^n$

combination $\rightarrow \binom{n}{r}=\frac{n!}{(n-r)!r!}$ $\quad$ binomial coefficients

Binomial Series used to Expand Power Series

$$(1+x)^k=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\dots \quad \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

18) $f(x)=\frac{1}{(1+x)^4}=(1+x)^{-4}$

$$=1-4x+\frac{-4(-5)}{2!}x^2-\frac{4(5)(6)}{3!}x^3+\frac{4\cdot5\cdot6\cdot7}{4!}x^4=1-4x+10x^2-20x^3+35x^4-\dots \sum_{n=0}^{\infty}(-1)^n\frac{(n+3)!}{3!n!}x^n$$

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20) $f(x)=\frac{1}{\sqrt{1-x^2}}=(1-x^2)^{-1/2}$

$$(1-x^2)^{-1/2}=1-\frac{1}{2}x^2+\frac{(\frac{1}{2})(\frac{3}{2})}{2!}x^4-\frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3!}x^6+\dots=1-\frac{1}{2}x^2+\frac{(1)(3)}{2^2 2!}x^4-\frac{(1)(3)(5)}{2^3(3!)}x^6+\dots$$

$$=1+\sum\frac{1\cdot3\cdot5\cdots(2n-1)}{2^n n!}x^{2n}$$

22) $f(x)=\frac{1}{(2+x)^3}=(2+x)^{-3}=\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$ $\quad k=-3$

$$=\frac{1}{8}\left[1-3\left(\frac{x}{2}\right)+\frac{3(4)}{2!}\left(\frac{x}{2}\right)^2-\frac{3(4)(5)}{3!}\left(\frac{x}{2}\right)^3\cdots\right]=\frac{1}{8}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+2)!}{2^{n+1}n!}x^n\right]$$

24) $f(x)=\sqrt[4]{1+x}=(1+x)^{1/4}$

$$=1+\frac{1}{4}x+\frac{(1/4)(-3/4)}{2!}x^2+\frac{(1/4)(-3/4)(-7/4)}{3!}x^3+\dots$$

$$=1+\frac{1}{4}x-\frac{3}{4^2\cdot 2!}x^2+\frac{3\cdot 7}{4^3\cdot 3!}x^3-\frac{3\cdot 7\cdot 11}{4^4\cdot 4!}x^4$$

$$=1+\frac{1}{4}x+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\cdots(2n-3)x^n}{4^n n!}$$

26) $f(x)=\sqrt{1+x^3}=(1+x^3)^{1/2}$

$$=1+\frac{x^3}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}1\cdot 3\cdot 5\dots(2n-3)x^{3n}}{2^n\cdot n!}$$

28) Find Maclaurin Series

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\qquad e^{-3x}=\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}=\sum\frac{(-1)^n 3^n x^n}{n!}=1-3x+\frac{9x^2}{2!}-\frac{27x^3}{3!}+\frac{81x^4}{4!}-\frac{243x^5}{5!}$$

30) $\ln x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-1)^n}{n}\quad 0<x\le 2$

$$\ln(x^2+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2+1-1)^n}{n}=\boxed{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}}$$

32) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n+1}}{(2n+1)!}$$

34) $$\cos x = \sum_{n=0}^{\infty} \frac{(-1) x^{2n}}{(2n)!} \qquad \cos(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!}$$

36) $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad 2\sin(x^3) = 2\sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$2\sin x^3 = 2\left[x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!}\right] = \boxed{2x^3 - \frac{2x^9}{3!} + \frac{x^{15}}{5!} - \cdots}$$

38) $$f(x) = e^x + e^{-x} = 2\cosh x$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots$$

$$2\cosh(x) = \sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!}$$

40) $$f(x) = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$(1+x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5 \cdots (2n-1) x^{2n}}{2^n n!}$$

$$\ln\left(x+\sqrt{x^2+1}\right) = \int \frac{1}{\sqrt{x^2+1}}\,dx = x + \cdots \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdot 5 \cdots (2n-1) x^{2n+1}}{2^n (2n+1) n!}$$

$$= x - \frac{x^3}{2\cdot 3} + \frac{1\cdot 3 x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5 x^7}{2\cdot 4\cdot 6\cdot 7}$$