2017

9.3 Integral Test and p-series Test p. 609 #1-38 0252

Determine convergence/divergence of series

* **Integral Test**: If f is positive, continuous, decreasing for $x \geq 1$ then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or diverge

* **P-series Test**: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $0 < p \leq 1$

2) $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ $f(x) = \frac{2}{3x+5}$ → positive, decreasing, continuous for $x \geq 1$

$\int_1^{\infty} \frac{2}{3x+5}dx$ $u = 3x+5$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$

$\lim_{b\to\infty} \int_1^b \frac{2}{u} \cdot \frac{du}{3} = \frac{2}{3}\ln|u| = \frac{2}{3}\ln|3x+5| \Big]_1^b =$

$\lim_{b\to\infty} \frac{2}{3}\ln|3b+5| - \frac{2}{3}\ln(3+5|) = \infty$

since $\int_1^{\infty} \frac{2}{3x+5}dx$ diverges, $\sum \frac{2}{3n+5}$ also diverges

4) $\sum_{n=1}^{\infty} 3^{-n} = \sum \left(\frac{1}{3}\right)^n$ let $f(x) = \frac{1}{3^x}$ f is positive, continuous, decreasing for $x \geq 1$

$\int_1^{\infty} \frac{1}{3^x}dx = \frac{-1}{(\ln 3)3^x}\Big]_1^{\infty} = \frac{1}{3\ln 3}$, so series also converge

6) $\sum_{n=1}^{\infty} ne^{-n/2}$ $f(x) = xe^{-x/2}$ $f(x)$ is positive, decreasing, continuous for $x \geq 3$

$f'(x) = \frac{e^{x/2} - xe^{x/2}(1/2)}{e^x} = \frac{e^{x/2}(1-x/2)}{e^x}$ ↗ 2 ↘

LIPET

u	dv
x	$e^{-x/2}$
1	$-2e^{-x/2}$
0	$4e^{-x/2}$

$-x\,2e^{-x/2} - 4e^{-x/2}\Big]_3^{\infty} = \frac{-2x}{e^{x/2}} - \frac{4}{e^{x/2}}\Big]_3^{\infty} = (0-0) - \left(\frac{-6}{e^{3/2}} - \frac{4}{e^{3/2}}\right)$

$= \frac{10}{e^{3/2}} = \boxed{\frac{10}{e^{3/2}}}$ series converges.

$\boxed{9.3}$

8) $\underbrace{\frac{1}{3}+\frac{1}{5}}_{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\ldots$

$a_n = a_1 + d(n-1)$

$a_n = 3+2(n-1)$

$a_n = 3+2n-2$

$\underline{a_n = 2n+1}$

$\sum_{n=1}^{\infty} \frac{1}{2n+1}$ Let $f(x) = \frac{1}{2x+1}$, $f'(x) = \frac{-2}{(2x+1)^2}$

f is positive, continuous, decreasing for $x \geq 1$

$\int_1^{\infty} \frac{1}{2x+1}dx$ $u = 2x+1$, $\frac{du}{dx} = 2$ | $dx = \frac{du}{2}$ $\int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|2x+1|\Big]_1^b = \frac{1}{2}\ln|2b+1| - \frac{1}{2}\ln|2(1)+1|$

$= \infty$

$\boxed{\text{diverges}}$

(by Integral Test)

$\frac{\ln 2}{\sqrt{2}} + \frac{\ln 3}{\sqrt{3}} + \frac{\ln 4}{\sqrt{4}} + \ldots \frac{\ln n}{\sqrt{n}}$

10) $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ $f(x) = \frac{\ln x}{\sqrt{x}}$ $f'(x) = \frac{2-\ln x}{2x^{3/2}}$ $f(x)$ is positive, continuous, decreasing for $x > e^2$

$\int_2^{\infty} \frac{\ln x}{\sqrt{x}}dx = \int (\ln x)(x^{-1/2})dx$

IBP Integration by Parts $u = \ln x$ $dv = x^{-1/2}$

$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{x^{1/2}}{1/2} = 2x^{1/2}$

$uv - \int v\,du$

$2x^{1/2}\ln x - \int 2\sqrt{x}\cdot\frac{1}{x}dx$ $= \int 2x^{-1/2} = \frac{2x^{1/2}}{1/2}$

$2\sqrt{x}\ln x - 4\sqrt{x}\Big]_2^{\infty} = \infty$ series diverge by Integral Test.

12) $\frac{1}{4}+\frac{2}{7}+\frac{3}{12}+\ldots+\ldots+\frac{n}{n^2+3}$ $\sum_{n=1}^{\infty}\frac{n}{n^2+3}$

$f(x) = \frac{x}{x^2+3}$ $f'(x) = \frac{1(x^2+3)-x(2x)}{(x^2+3)^2} = \frac{-x^2+3}{(x^2+3)^2}$ $\sqrt{3}$ $f(x)$ is positive, continuous, decreasing for $x \geq 2$

$\int_2^{\infty}\frac{x}{x^2+3}dx$ $u = x^2+3$ $\frac{du}{dx} = 2x$ $dx = \frac{du}{2x}$

$\int \frac{x}{u}\cdot\frac{du}{2\cancel{x}} = \frac{1}{2}\ln|x^2+3|\Big]_2^{\infty} = \infty$, so series diverge by Integral Test.

$\boxed{9.3}$

14) $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$ let $f(x) = \frac{\ln x}{x^3}$ $f'(x) = \frac{1-3\ln x}{x^4}$ f(x) is positive, continuous, decreasing for $x > 2$

$\int_2^{\infty} \frac{\ln x}{x^3} dx$ $\int (\ln x)(x^{-3}) dx$ | IBP $u = \ln x$ $dv = x^{-3}$ $du = \frac{1}{x} dx$ $v = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$ | $uv - \int v\,du$ $\frac{-\ln x}{2x^2} - \int \frac{-1}{2x^2}\left(\frac{1}{x}\right) dx = \int \frac{1}{2x^3}$ $\frac{-\ln x}{2x^2} + \frac{1}{2}\int x^{-3} dx$

$= \frac{-\ln x}{2x^2} + \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) = \frac{-\ln x}{2x^2} - \frac{1}{4x^2} = \left.\frac{-2\ln x - 1}{4x^2}\right]_2^{\infty}$

$\overset{\nearrow 0}{\frac{-2\ln(\infty - 1)}{4\infty^2}} - \left[\frac{-2\ln 2 - 1}{4(2)^2}\right] = \boxed{\frac{2\ln 2 + 1}{16}}$, series converges by Integral Test

16) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ $f(x) = \frac{1}{x\sqrt{\ln x}}$ $f'(x) = \frac{-2\ln x + 1}{2x^2(\ln x)^{3/2}}$ f is positive, continuous, decreasing for $x \geq 2$.

$\int_2^{\infty} \frac{1}{x(\ln x)^{1/2}} dx$ $u = \ln x$ $\frac{du}{dx} = \frac{1}{x}$ | $dx = x\,du$ $\int \frac{1}{\cancel{x} \cdot u^{1/2}} \cancel{x} du = \int u^{-1/2} du$ $\frac{u^{1/2}}{1/2} = \left. 2(\ln x)^{1/2} \right]_2^{\infty} = \infty$ diverges by Integral Test.

18) $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ $f(x) = \frac{x+2}{x+1}$ $f'(x) = \frac{-1}{(x+1)^2} < 0$ f(x) is positive, continuous decreasing

$\int \frac{x+2}{x+1} dx$ $\begin{array}{r} 1 + \frac{1}{x+1} \\ x+1 \overline{) x+2} \\ \ominus x \overset{\ominus}{+} 1 \\ \hline 1 \end{array}$ $= \int 1 + \frac{1}{x+1} dx = \left. x + \ln|x+1| \right]_1^{\infty} = \infty$ diverges by Integral Test.

20) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ $f(x) = \frac{1}{\sqrt{x+2}}$ $f'(x) = \frac{-1}{2(x+2)^{3/2}} < 0$ f(x) is pos, contin, dec. for $x \geq 1$

$\int \frac{1}{\sqrt{x+2}} dx$ $u = x+2$ $\frac{du}{dx} = 1$ $\int u^{-1/2} du = \frac{u^{1/2}}{1/2} = \left. 2\sqrt{x+2} \right]_1^{\infty} = \infty$ diverges by Integral Test.

22) $\sum_{n=1}^{\infty} \frac{n}{n^4+2n^2+1} = \frac{n}{(n^2+1)^2}$ $\quad f(x) = \frac{x}{(x^2+1)^2}$ $\quad f'(x) = \frac{-3x^2+1}{(x^2+1)^3} < 0$ for $x \geq 1$

f is pos, continuous, decreasing for $x \geq 1$

$\int_1^{\infty} \frac{x}{(x^2+1)^2} dx$ $\quad u = x^2+1$ $\quad \frac{du}{dx} = 2x$ $\quad \Big| \quad dx = \frac{du}{2x}$

$\int \frac{x}{u^2} \cdot \frac{du}{2\cancel{x}} = \frac{1}{2}\int u^{-2} = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) = -\frac{1}{2(x^2+1)}\Big]_1^{\infty} = 0 - \left(\frac{-1}{2(2)}\right)$

$= \boxed{1/4}$

series converge by Integral Test.

26) $\sum_{n=1}^{\infty} e^{-n} \cos n$ Integral Test does not apply since $f(x)$ not positive for $x \geq 1$

28) $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^2 \rightarrow$ Integral Test does not apply since $f(x)$ not decreasing for $x \geq 1$

30) Apply Integral Test

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ $\quad f(x) = \frac{1}{\sqrt{x}}$ $\quad f(x)$ pos, continuous, decreasing for $x \geq 1$

$\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}\Big]_1^{\infty} = \infty$, diverges by Integral Test

32) $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $\quad f(x) = \frac{1}{x^5}$ $\quad f(x)$ pos, continuous, dec for $x \geq 1$

$\int x^{-5} dx = \frac{x^{-4}}{-4} = \frac{-1}{4x^4}\Big]_1^{\infty} = 0 - \left(\frac{-1}{4}\right) = \boxed{1/4}$ converges by Integral Test.

p-series

34) $\sum_{n=1}^{\infty} \frac{3}{n^{5/3}}$ $p = 5/3 > 1$, series converges

36) $1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \dots \frac{1}{\sqrt[3]{n^2}}$ $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p = 2/3 < 1$
divergent
p-series.

38) $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ $p = \pi > 1$ convergent p-series.