Ch. 9.4 Direct Comparison Test (DCT) and Limit Comparison Test (LCT)

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<u>Direct Comparison Test</u>: Use this test to determine convergence/divergence for more complex, positive series by comparing with simpler series.

let $0 < a_n < b_n$

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

<u>Use Direct Comparison Test</u>

4) $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ Since $\frac{1}{3n^2+2} < \frac{1}{3n^2}$. $\frac{1}{3n^2}$ converges by p-series test, then by DCT, $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ also converges.

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6) $\sum_{n=0}^{\infty} \frac{4^n}{5^n+3}$ $\frac{4^n}{5^n+3} < \left(\frac{4}{5}\right)^n$ series converge by DCT since $\left(\frac{4}{5}\right)^n$ converges by Geometric Series Test $r = 4/5 < 1$

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8) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ $\frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$ series converge by DCT since $\frac{1}{n^{3/2}}$ is a convergent p-series $(p = 3/2 > 1)$

10) $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ $\frac{1}{4\sqrt[3]{n}-1} > \frac{1}{\sqrt[3]{n}}$ series diverge by DCT since $\frac{1}{n^{1/3}}$ is a divergent p-series $(p = 1/3 \leq 1)$

12) $\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$ $\frac{3^n}{2^n-1} > \left(\frac{3}{2}\right)^n$ series diverge by DCT since $\left(\frac{3}{2}\right)^n$ is a divergent Geometric series $r = \left|\frac{3}{2}\right| > 1$

series in the problem

simpler series for test case

Limit Comparison Test

* Given $a_n > 0$ and $b_n > 0$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) \text{ or } \lim_{n\to\infty}\left(\frac{b_n}{a_n}\right) = L$$ and L is finite and positive,

then both series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Use Limit Comparison Test

14) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{4^n+1}$

$a_n = \dfrac{5}{4^n+1}$

$b_n = \dfrac{1}{4^n}$ (convergent geometric series $r = 1/4 < 1$)

$$\lim_{n\to\infty} \frac{\frac{5}{4^n+1}}{\frac{1}{4^n}} = \lim_{n\to\infty} \frac{5}{4^n+1}\cdot\frac{4^n}{1} = 5$$

finite, positive

Therefore by LCT, series converge

16) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{5^n+1}$

$a_n = \dfrac{2^n+1}{5^n+1}$

$b_n = \dfrac{2^n}{5^n}$ (convergent geometric series $r = \left|\frac{2}{5}\right| < 1$)

$$\lim_{n\to\infty} \frac{\frac{2^n+1}{5^n+1}}{\frac{2^n}{5^n}} = \lim_{n\to\infty} \frac{2^n+1}{5^n+1}\cdot\frac{5^n}{2^n} = 1$$

Series converge by LCT.

18) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$

$a_n = \dfrac{1}{n^2(n+3)}$

$b_n = \dfrac{1}{n^3}$ (convergent p-series)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2(n+3)}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{n^2(n+3)}\cdot\frac{n^3}{1} = 1$$

series converge by LCT

20) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}} = \frac{2n}{(n+1)2^n}$

$a_n = \dfrac{2n}{(n+1)2^n}$

$b_n = \dfrac{1}{2^n}$ (convergent geometric series)

$$\lim_{n\to\infty} \frac{\frac{2n}{(n+1)2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2n}{(n+1)\cancel{2^n}}\cdot\frac{\cancel{2^n}}{1} = 2$$

series converge by LCT

* Recall L'Hopital's Rule

22) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

$a_n = \sin\left(\frac{1}{n}\right)$;

$b_n = \frac{1}{n}$ (divergent p-series, harmonic series)

$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \xrightarrow{L'H} \frac{-n^{-2}\cdot\cos(1/n)}{-n^{-2}}$

$= \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$

Series Diverges by LCT.

Determine Convergence/divergence using appropriate tests:

a) n^{th} term test
b) p-series
c) Integral Test
d) Geometric Series test
e) Telescoping Series Test
f) DCT
g) LCT

24) $\sum_{n=0}^{\infty} 5\left(\frac{-4}{3}\right)^n$ Geometric Series Test: $r = \left|\frac{-4}{3}\right| > 1$, divergent series:
* can also use LCT, DCT

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26) $\sum_{n=2}^{\infty} \frac{1}{n^3-8}$

$a_n = \frac{1}{n^3-8}$

$b_n = \frac{1}{n^3}$ (convergent p-series)

$\lim_{n\to\infty} \frac{\frac{1}{n^3-8}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{1}{n^3-8}\cdot\frac{n^3}{1} = 1$

converges by LCT

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28) $\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+2} = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\dots = \frac{1}{2}$ converges by Telescoping series.

29) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ Let $f(x) = \frac{x}{(x^2+1)^2}$ $f(x)$ is pos, continuous, decreasing for $x \geq 1$

$\int_1^{\infty} \frac{x}{(x^2+1)^2}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$\int \frac{x}{u^2}\cdot\frac{du}{2x}$

$\frac{1}{2}\int u^{-2}du =$

$\frac{1}{2}\left(\frac{u^{-1}}{-1}\right) = \left.\frac{-1}{2(x^2+1)}\right]_1^{\infty} = \frac{-1}{\infty} - \left(\frac{-1}{2(2)}\right) = \boxed{1/4}$

converges by Integral Test.

23) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$

let $a_n = \dfrac{\sqrt[3]{n}}{n}$

$b_n = \dfrac{1}{n^{2/3}}$ (divergent p-series $p = 2/3 < 1$)

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n}}{n}}{\frac{1}{n^{2/3}}}$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n} \cdot \frac{n^{2/3}}{1} = 1$$

Diverges by LCT.

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25) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ $\qquad \dfrac{1}{5^n+1} < \dfrac{1}{5^n}$

Since $\frac{1}{5^n}$ converges by Geometric series test $r = \left|\frac{1}{5}\right| < 1$

series converge by DCT.

27) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n-2}$

$\cancel{a_n = \dfrac{2^n}{3^n-2}}$

$\cancel{b_n = \dfrac{2^n}{3^n}}$ ~~(converges by Geometric Series test $r = \left|\frac{2}{3}\right| < 1$)~~

$$\cancel{\lim_{n\to\infty} \frac{\frac{2^n}{3^n-2}}{\frac{2^n}{3^n}} = \lim_{n\to\infty} \frac{2^n}{3^n-2} \cdot \frac{3^n}{2^n} = 1}$$

$$\lim_{n\to\infty} \frac{2^n}{3^n-2} = \frac{2}{3} \neq 0$$ Diverges by n^{th} term test.

30) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{1}{n} - \frac{1}{n+3}$

$A = 1 \quad B = -1$

$n = 0 \quad n = -3$

$$= \left(1 - \tfrac{1}{4}\right) + \left(\tfrac{1}{2} - \tfrac{1}{5}\right) + \left(\tfrac{1}{3} - \tfrac{1}{6}\right) + \left(\tfrac{1}{4} - \tfrac{1}{7}\right) + \left(\tfrac{1}{5} - \tfrac{1}{8}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{6+3+2}{6} = \boxed{\frac{11}{6}}$$

Converges by Telescoping Series Test.