9.6 Ratio Test and Root Test p. 633 13–72

***Ratio Test:**

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$
2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$
3) Ratio Test inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

14) $\sum_{n=1}^{\infty}\frac{1}{n!}$ $\quad \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{1}{(n+1)!}\cdot n! = \lim_{n\to\infty}\frac{\cancel{n!}}{(n+1)\cdot\cancel{n!}} = \frac{1}{n+1} = 0 < 1$

series converge by Ratio Test

16) $\sum_{n=0}^{\infty}\frac{2^n}{n!}$ $\quad \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2\cdot\cancel{n!}}{(n+1)\cdot\cancel{n!}} = 0 < 1$

series converge by Ratio Test

18) $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$ $\quad \lim_{n\to\infty}\frac{(n+1)(7/8)^{n+1}}{n\cdot(7/8)^n} = \lim_{n\to\infty} 7/8 = 7/8 < 1$ series converge by Ratio Test

20) $\sum_{n=1}^{\infty}\frac{5^n}{n^4}$ $\quad \lim_{n\to\infty}\frac{5^{n+1}}{(n+1)^4}\cdot\frac{n^4}{5^n} = \lim_{n\to\infty}\frac{\cancel{5^n}\cdot 5\cdot n^4}{\cancel{5^n}\cdot(n+1)^4} = 5 > 1$ diverge by Ratio Test (or n^{th} term test)

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22) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$ $\quad \lim_{n\to\infty}\frac{n+3}{(n+1)(n+2)}\cdot\frac{n(n+1)}{n+2} = 1$ inconclusive by Ratio Test

$\lim_{n\to\infty}\frac{n+2}{n(n+1)} = 0$ converges (conditionally) by Alt. Series Test.

24) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2}$ $\quad \lim_{n\to\infty}\frac{\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{\left(\frac{3}{2}\right)^n} = \lim_{n\to\infty}\frac{n^2\cdot(3/2)}{(n+1)^2} = \frac{3}{2} > 1$ series diverge by Ratio Test.

26) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$ $\lim_{n\to\infty} \frac{\overset{(2n+2)!}{(2(n+1))!}}{(n+1)^5} \cdot \frac{n^5}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)!}\, n^5}{(n+1)^5 \cancel{(2n)!}} = \infty > 1$

Series diverge by Ratio Test.

28) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ $\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)}\cancel{(n!)}}{(n+1)^n \cdot \cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}}$ $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$

converges by Ratio Test

30) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$ $\lim_{n\to\infty} \frac{((n+1)!)^2}{[3(n+1)]!} \cdot \frac{(3n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot \cancel{n!^2}\, \cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}\, \cancel{(n!)^2}} = 0 < 1$

Series converge by Ratio Test.

32) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$ $\lim_{n\to\infty} \frac{2^{4(n+1)}}{\underset{(2n+3)!}{[2(n+1)+1]!}} \cdot \frac{(2n+1)!}{2^{4n}} = \lim_{n\to\infty} \frac{2^{4n+4}\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}\, 2^{4n}}$

$= \lim_{n\to\infty} \frac{2^4}{(2n+3)(2n+2)} = 0 < 1$, converges by Ratio Test

34) $\sum_{n=1}^{\infty} \frac{(-1)^n [2\cdot 4\cdot 6 \cdot \ldots (2n)]}{2\cdot 5\cdot 8 \cdot \ldots (3n-1)}$ $\lim_{n\to\infty} \frac{\cancel{2\cdot 4\cdot 6 \cdots (2n)}(2n+2)}{\cancel{2\cdot 5\cdot 8 \cdots (3n-1)}(3n+2)} \cdot \frac{\cancel{2\cdot 5\cdot 8 \cdots (3n-1)}}{\cancel{2\cdot 4\cdot 6 \cdots (2n)}}$

$\lim_{n\to\infty} \frac{2n+2}{3n+2} = \frac{2}{3} \neq 1$ Series converge by Ratio Test

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* Root Test: (well-suited for series involving n^{th} powers)

1) $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2) $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$
3) Root test inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

36) $\sum\limits_{n=1}^{\infty} \frac{1}{n^n}$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{1}{n}\right)^n} = 0 < 1$, converges by Root Test

38) $\sum\limits_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^n} = 2 > 1$ diverges by Root Test

40) $\sum\limits_{n=1}^{\infty} \left(\frac{n-2}{5n+1}\right)^n$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n-2}{5n+1}\right)^n} = \frac{1}{5} < 1$, converges by Root Test

42) $\sum\limits_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left|\frac{-3n}{2n+1}\right|^{3n}} = \lim\limits_{n\to\infty} \left|\frac{-3n}{2n+1}\right|^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8} > 1$ series diverge by Root Test.

44) $\sum\limits_{n=0}^{\infty} e^{-3n} = \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{1}{e^3}\right)^n} = \lim\limits_{n\to\infty} \frac{1}{e^3} < 1$ converge by Root test

46) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n}{500}\right)^n}$ $\quad \lim\limits_{n\to\infty} \frac{n}{500} = \infty$, diverges by Root test

48) $\sum\limits_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{\ln n}{n}\right)^n}$ $\quad \lim\limits_{n\to\infty} \frac{\ln n}{n} \xrightarrow{L'H} \frac{\frac{1}{n}}{1} = 0 < 1$ converge by Root test.

50) $\sum\limits_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$ $\quad \lim\limits_{n\to\infty} \sqrt[n]{\left(\frac{n!}{n^2}\right)^n}$ $\quad \lim\limits_{n\to\infty} \frac{n!}{n^2} = \infty > 1$ diverges by Root test

Determine Convergence/divergence. Identify convergence test used.

52) $\sum_{n=1}^{\infty} \frac{100}{n} = 100 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test (harmonic series $p=1$)

54) $\sum_{n=1}^{\infty} \left(\frac{2\pi}{3}\right)^n$ $r = \left|\frac{2\pi}{3}\right| > 1$, diverges by Geometric Series Test

56) $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

let $a_n = \frac{n}{2n^2+1}$

$b_n = \frac{1}{n}$ (divergent p-series)

$\lim_{n\to\infty} \frac{\frac{n}{2n^2+1}}{\frac{1}{n}}$

$\lim_{n\to\infty} \frac{n}{2n^2+1} \cdot \frac{n}{1} = \frac{1}{2}$ (Diverges by Limit Comparison Test)

58) $\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$ $\frac{10}{3} \sum \frac{1}{n^{3/2}}$ convergent p-series $(p = 3/2 > 1)$

60) $\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$ $\lim_{n\to\infty} \frac{2^n}{4n^2-1} = \infty \neq 0$ Diverges by n^{th} term test

62) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ Converges by Alt. Series Test.

$a_{n+1} \leq a_n$

64) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ $\frac{\ln n}{n^2} \leq \frac{1}{n^2}$ (convergent p-series) | series converge by Direct Comparison Test.

66) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n 2^n}$ $\lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{3^n}\right| = \lim_{n\to\infty} \left|\frac{3}{2} \cdot \frac{n}{n+1}\right| = \frac{3}{2} > 1$ Diverges by Ratio Test

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68) $\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7 \cdots (2n+1)}{18^n(2n-1)\,n!}$ $\lim_{n\to\infty} \frac{\cancel{3\cdot 5\cdot 7}\cdots \overset{(2n+1)}{}(2n+3)}{18^{n+1}(2n+1)(2n-1)(n+1)!} \cdot \frac{18^n(2n-1)n!}{\cancel{3\cdot 5\cdot 7}\cdots \cancel{(2n+1)}}$

$\lim_{n\to\infty} \frac{(2n+3)(2n-1)\cancel{n!}}{18\cdot(2n+1)(2n-1)\cancel{n!}}$ $\lim_{n\to\infty} \frac{4n^2}{72n^2} = \frac{1}{18} < 1$ converges by Ratio Test.

70) Identify the same series

a) $\sum_{n=4}^{\infty} n\left(\frac{3}{4}\right)^n$

same [b) $\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n$

c) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$]

72) Identify the same series

same [a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$]

c) $\sum_{n=0} \frac{(-1)^{n+1}}{(n+1)2^n}$