9.7a Taylor Polynomials p.644 12-43

Maclaurin Series : $F(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$ (centered at c=0)

Taylor Series: $F(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!} (x-c)^n$

12) Find Maclaurin polynomial P_2, P_3, P_4 for $f(x) = x^2 e^x$ $f(0)=0$

$f'(x) = 2xe^x + x^2 \cdot e^x = (x^2+2x)e^x$ $f'(0)=0$

$f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2+4x+2)e^x$ $f''(0)=2$

$f'''(x) = (x^2+6x+6)e^x$ $f'''(0)=6$

$f^4(x) = (x^2+8x+12)e^x$ $f^4(0)=12$

$P_2(x) = 0 + 0(x-0) + \frac{2}{2!}(x-0)^2 = x^2$

$P_3(x) = x^2 + \frac{6}{3!}(x-0)^3 = x^2 + x^3$

$P_4(x) = x^2 + x^3 + \frac{12}{4!}(x-0)^4 = x^2 + x^3 + \frac{1}{2}x^4$

Find the Maclaurin polynomial of degree n for the function:

14) $f(x) = e^{-x}$ $n=5$ $f(0)=1$ $P_5(x) = \underbrace{f(0) + f'(0)x}_{\text{tangent line}} + \frac{f^n(0)}{n!}(x-0)^n$

$f'(x) = -e^{-x}$ $f'(0) = -1$

$f''(x) = e^{-x}$ $f''(0) = 1$ $P_5(x) = 1 + -1x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5$

$f'''(x) = -e^{-x}$ $f'''(0) = -1$

$f^4(x) = e^{-x}$ $f^4(0) = 1$ $\boxed{P_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}}$

$f^5(x) = -e^{-x}$ $f^5(0) = -1$

16) $f(x) = e^{x/3}$ $\quad n=4$

$f(0)=1$

$f'(x) = \frac{1}{3}e^{x/3} \quad f'(0) = \frac{1}{3}$

$f''(x) = \frac{1}{9}e^{x/3} \quad f''(0) = \frac{1}{9}$

$f'''(x) = \frac{1}{27}e^{x/3} \quad f'''(0) = \frac{1}{27}$

$f^4(x) = \frac{1}{81}e^{x/3} \quad f^4(0) = \frac{1}{81}$

$P_4(x) = 1 + \frac{1}{3}x + \frac{1/9}{2!}x^2 + \frac{1/27}{3!}x^3 + \frac{1/81}{4!}x^4$

$P_4(x) = 1 + \frac{1}{3}x + \frac{1}{18}x^2 + \frac{1}{162}x^3 + \frac{1}{1944}x^4$

18) $f(x) = \cos \pi x$ $\quad n=4$

$f(0) = 1$

$f'(x) = -\pi \sin \pi x \quad f'(0) = 0$

$f''(x) = -\pi^2 \cos \pi x \quad f''(0) = -\pi^2$

$f'''(x) = \pi^3 \sin \pi x \quad f'''(0) = 0$

$f^4(x) = \pi^4 \cos \pi x \quad f^4(0) = \pi^4$

$P_4(x) = 1 + 0x + \frac{-\pi^2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{\pi^4}{4!}x^4$

$P_4(x) = 1 - \frac{\pi^2}{2}x^2 + \frac{\pi^4}{24}x^4$

20) $f(x) = x^2 e^{-x}$ $\quad n=4$

$f(0) = 0$

$f'(x) = 2xe^{-x} + x^2e^{-x}(-1) \quad f'(0) = 0$

$f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} \quad f''(0) = 2$

$f'''(x) = -6e^{-x} + 6xe^{-x} - x^2e^{-x} \quad f'''(0) = -6$

$f^4(x) = 12e^{-x} - 8xe^{-x} + x^2e^{-x} \quad f^4(0) = 12$

$P_4(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{-6}{3!}x^3 + \frac{12}{4!}x^4$

$P_4(x) = x^2 - x^3 + \frac{1}{2}x^4$

42) Approximate function using polynomial from #20

$f(\frac{1}{5}) \approx P_4(\frac{1}{5}) = (\frac{1}{5})^2 - (\frac{1}{5})^3 + \frac{1}{2}(\frac{1}{5})^4 = \boxed{0.0328}$

22) $f(x) = \frac{x}{x+1}$ $f(0)=0$ $n=4$

$f'(x) = \frac{\overbrace{(1)(x+1) - x(1)}^{x+1-x}}{(x+1)^2} = \frac{1}{(x+1)^2} = (x+1)^{-2}$ $f'(0)=1$

$f''(x) = -2(x+1)^{-3}$ $f''(0) = -2$

$f'''(x) = 6(x+1)^{-4}$ $f'''(0) = 6$

$f^4(x) = -24(x+1)^{-5}$ $f^4(0) = -24$

$P_4(x) = 0 + 1x - \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{-24}{4!}x^4$

$P_4(x) = x - x^2 + x^3 - x^4$

24) $f(x) = \tan x$ $f(0)=0$ $n=3$

$f'(x) = \sec^2 x = [\sec x]^2$ $f'(0)=1$

$f''(x) = 2[\sec x]\sec x \tan x = 2\sec^2 x \tan x$ $f''(0)=0$

$f'''(x) = 4\sec x(\sec x \tan x)\tan x + 2\sec^2 x(\sec^2 x)$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$ $f'''(0)=2$

$P_3(x) = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3$

$P_3(x) = x + \frac{1}{3}x^3$

Find a Taylor Polynomial $F(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n$

26) $f(x) = \frac{1}{x^2}$ $n=4, c=2$

$f(x) = x^{-2}$ $f(2) = \frac{1}{4}$

$f'(x) = -2x^{-3}$ $f'(2) = -1/4$

$f''(x) = 6x^{-4}$ $f''(2) = 3/8$

$f'''(x) = -24x^{-5}$ $f'''(2) = -3/4$

$f^4(x) = 120x^{-6}$ $f^4(2) = 15/8$

$P_4(x) = \frac{1}{4} + \frac{-1}{4}(x-2) + \frac{3/8}{2!}(x-2)^2 + \frac{-3/4}{3!}(x-2)^3 + \frac{15/8}{4!}(x-2)^4$

$P_4(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{5}{64}(x-2)^4$

30) $f(x)=x^2\cos x$ $\quad n=2 \quad c=\pi$

$f(\pi)=-\pi^2$

$f'(x)=2x\cos x+x^2(-\sin x)$ $\quad f'(\pi)=-2\pi+0$

$f''(x)=2\cos x+2x(-\sin x)+2x(-\sin x)+x^2(-\cos x)$
$=2\cos x-4x\sin x-x^2\cos x$ $\quad f''(\pi)=-2+\pi^2$

$P_2(x)=-\pi^2+(-2\pi)(x-\pi)$
$+\dfrac{\pi^2-2}{2!}(x-\pi)^2$

32) Find Taylor Polynomial $f(x)=\dfrac{1}{x^2+1}=(x^2+1)^{-1}$

a) $n=4$, $c=0$ (Maclaurin polynomial)

$f(x)=(x^2+1)^{-1}$ $\quad \underline{f(0)=1}$

$f'(x)=-1(x^2+1)^{-2}(2x)=-2x(x^2+1)^{-2}$ $\quad \underline{f'(0)=0}$

$f''(x)=-2(x^2+1)^{-2}+(-2x)\cdot-2(x^2+1)^{-3}(2x)=-2(x^2+1)^{-2}+8x^2(x^2+1)^{-3}$ $\quad \underline{f''(0)=-2}$

$f'''(x)=4(x^2+1)^{-3}(2x)+16x(x^2+1)^{-3}+8x^2\cdot-3(x^2+1)^{-4}(2x)$ $\quad \underline{f'''(0)=0}$
$=8x(x^2+1)^{-3}+16x(x^2+1)^{-3}-48x^3(x^2+1)^{-4}$

$f^4(x)=8(x^2+1)^{-3}+8x\cdot-3(x^2+1)^{-4}(2x)+16(x^2+1)^{-3}+16x\cdot-3(x^2+1)^{-4}(2x)+-144x^2(x^2+1)^{-4}$
$+-48x^3\cdot-4(x^2+1)^{-5}(2x)$

$\underline{f^4(0)=8+16=24}$

$P_4(x)=1+0x+\dfrac{-2}{2!}x^2+\dfrac{0}{3!}x^3+\dfrac{24}{4!}x^4$

$\boxed{P_4(x)=1-x^2+x^4}$

b) $n=4 \quad c=1$

$f(1)=\frac{1}{2}$

$f'(1)=-2(2)^{-2}=\frac{-1}{2}$

$f''(1)=\frac{-1}{2}+1=\frac{1}{2}$

$f'''(1)=1+2-3=0$

$f^4(1)=1-3+2-6-9+12=-3$

$T_4(x)=\frac{1}{2}+\frac{-1}{2}(x-1)+\frac{1/2}{2!}(x-1)^2+\frac{0}{3!}(x-1)^3+\frac{-3}{4!}(x-1)^4$

$\boxed{T_4(x)=\frac{1}{2}-\frac{1}{2}(x-1)+\frac{1}{4}(x-1)^2-\frac{1}{8}(x-1)^4}$