2017

9.7b LaGrange Error p.645 #45-58

* LaGrange Error Bound $R_n(x) = \left|P_n(x) - f(x)\right| \le \dfrac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1}$

Use Taylor's Theorem to find upper bound for error, then calculate value of error.

45) $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

$R_4(x) \le \dfrac{\max\left|f^5(z)\right|}{5!}(x-c)^5$

0.3 0

$f^5(x) = -\sin x$ on $[0, 0.3]$ (c, x)

The max value of $-\sin x$ for $0 < z < 0.3$ is 1

$R_4(x) \le \dfrac{1}{5!}(0.3-0)^5 = \boxed{2.025\times10^{-5}}$

46) $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

$R_5(x) \le \dfrac{\max\left|f^6(z)\right|}{6!}(x-c)^6$

1 0

$f(x) = e^x$

$f^6(x) = e^x$ on $[0, 1]$

Max value of e^x on $0 < x < 1$ is e^1

$R_5(x) \le \dfrac{e^1}{6!}(1-0)^6 \approx \boxed{3.78\times10^{-3}}$

47) $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2\cdot 3}$

$f(x) = \arcsin x \qquad f'(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$f''(x) = -\frac{1}{2}(1-x^2)^{-3/2}(-2x) = +x(1-x^2)^{-3/2}$

$f'''(x) = 1(1-x^2)^{-3/2} + x\cdot -\frac{3}{2}(1-x^2)^{-5/2}(-2x)$

$= (1-x^2)^{-3/2} + 3x^2(1-x^2)^{-5/2}$

$f^4(x) = -\frac{3}{2}(1-x^2)^{-5/2}(-2x) + 6x(1-x^2)^{-5/2} + 3x^2\cdot -\frac{5}{2}(1-x^2)^{-7/2}(-2x)$

$= \dfrac{3x}{(1-x^2)^{5/2}} + \dfrac{6x}{(1-x^2)^{5/2}} + \dfrac{15x^3}{(1-x^2)^{7/2}} = \dfrac{9x(1-x^2)+15x^3}{(1-x^2)^{7/2}}$

$= \dfrac{6x^3+9x}{(1-x^2)^{7/2}}$

$R_3(x) \le \dfrac{\max\left|f^4(z)\right|}{4!}(x-c)^4$

0.4 0

Max value of $f^4(x)$ on $[0, 0.4]$ is $f^4(0.4)$
$= 7.3340$

$R_3(x) \le \dfrac{7.3340}{4!}(0.4-0)^4 \approx 0.00782$
$= \boxed{7.82\times10^{-3}}$

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48) $\arctan(0.4) \approx 0.4 - \dfrac{(0.4)^3}{3}$

$f(x) = \arctan x \quad f'(x) = \dfrac{1}{1+x^2} = (1+x^2)^{-1}$

$f''(x) = -1(1+x^2)^{-2}(2x) = -2x(1+x^2)^{-2}$

$f'''(x) = -2(1+x^2)^{-2} + (-2x)\cdot(-2)(1+x^2)^{-3}(2x)$

$= -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3}$

$f^4(x) = +4(1+x^2)^{-3}(2x) + 16x(1+x^2)^{-3} + 8x^2\cdot -3(1+x^2)^{-4}(2x)$

$= \dfrac{8x}{(1+x^2)^3} + \dfrac{16x}{(1+x^2)^3} - \dfrac{48x^3}{(1+x^2)^4} = \dfrac{24x(1+x^2)-48x^3}{(1+x^2)^4} = \dfrac{24x(x^2+1)}{(1+x^2)^4}$

$R_3(x) \le \dfrac{\text{Max}|f^4(z)|}{4!}(x-c)^4$ ($x \leftarrow 0.4$, $c \leftarrow 0$)

Max value of $f^4(x)$ on $[0, 0.4]$ is $f^4(0.4) \approx 22.3672$

$R_3(x) \le \dfrac{22.3672}{4!}(0.4-0)^4 \approx \boxed{0.0239}$

Determine degree of Maclaurin polynomial $(c=0)$ for error of approximation be less than 0.001

49) $\sin(0.3)$

$g(x) = \sin x$

$|g^{n+1}(x)| \le 1$ for all x

max value

$R_n(x) \le \dfrac{\max|g^{n+1}(z)|}{(n+1)!}(x-c)^{n+1} < 0.001$ ($x \leftarrow 0.3$, $c \leftarrow 0$)

$R_n(x) \le \dfrac{1}{(n+1)!}(0.3)^{n+1} < 0.001$

By Trial and error, $n=3$

50) $\cos(0.1)$

$g(x) = \cos x$

$|g^{n+1}(x)| \le 1$ for all x

$R_n(x) \le \dfrac{\max|g^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} < 0.001$ ($x \leftarrow 0.1$, $c \leftarrow 0$)

$R_n(x) \le \dfrac{1}{(n+1)!}(0.1)^{n+1} < 0.001$

By trial and error, $n=2$

51) $e^{0.6}$

$g(x) = e^x$

Max on $[0, 0.6]$ for e^x is $e^{0.6} \approx 1.8221$

$$R_n(x) \le \frac{\text{Max}|g^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} < 0.001$$

0.6 0

$$R_n(x) \le \frac{1.8221}{(n+1)!}(0.6)^{n+1} < 0.001$$

$\boxed{\text{By trial and error, } n = 5}$

52) $\ln(1.25)$

$g(x) = \ln x$

Max on $[1, 1.25]$ for $\ln x$ is $n!$

$g'(x) = \frac{1}{x} = x^{-1}$

$g''(x) = -1x^{-2}$

$g'''(x) = 2x^{-3}$

$g^4(x) = -6x^{-4}$

$g^5(x) = 24x^{-5}$

$f^{n+1}(x) = (-1)^n \frac{n!}{x^{n+1}}$

$$R_n(x) \le \frac{\text{Max}|g^{(n+1)}(z)|}{(n+1)!}(x-c)^{n+1} < 0.001$$

1.25 1

$$R_n(x) \le \frac{n!}{(n+1)!}(0.25)^{n+1} < 0.001$$

$$= \frac{1}{n+1}(0.25)^{n+1} < 0.001$$

$\boxed{\text{By trial and error, } n = 3}$

54) Determine degree of Maclaurin polynomial for error to be less than 0.0001

$f(x) = e^{-\pi x}$ approximate $f(1.3)$

$f'(x) = -\pi e^{-\pi x}$

$f''(x) = +\pi^2 e^{-\pi x}$

$f'''(x) = -\pi^3 e^{-\pi x}$

Max value $\le |(-\pi)^{n+1}|$ on $[0, 1.3]$

$$R_n(x) \le \frac{|\pi^{n+1}|}{(n+1)!}(x-c)^{n+1} < 0.0001$$

$$= \frac{\pi^{n+1}}{(n+1)!}(1.3-0)^{n+1} < 0.0001$$

$\boxed{\text{By trial and error, } n = 16}$

Determine values of x where error cannot exceed 0.001

55) $f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$; $x < 0$

$R_3(x) \leq \frac{e^z}{4!}(x-0)^4 < 0.001$, $z < 0$

$\frac{e^z}{4!}x^4 < 0.001$

$e^z x^4 = 0.024$

$(e^z x^4)^{1/4} = (0.024)^{1/4}$

$|e^{z/4} x| = 0.3936$

$|x| = \frac{0.3936}{e^{z/4}} < 0.3936$

$\boxed{-0.3936 < x < 0}$

56) $f(x) = \sin x \approx x - \frac{x^3}{3!}$

$R_n(x) \leq \frac{\max|\sin z|}{4!}(x-c)^4 \leq 0.001$

$\frac{1}{4!}x^4 < 0.001$

$x^4 = 0.024$

$|x| = 0.3936$

$\boxed{-0.3936 < x < 0.3936}$

57) $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

* first unused term is $n=6$

$R_4(x) \leq \frac{\max|\cos z|}{6!}(x-0)^6 < 0.001$

$= \frac{1}{6!}x^6 < 0.001$

$x^6 = 0.72$

$|x| < 0.9467$

$\boxed{-0.9467 < x < 0.9467}$

58) $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

$f'(x) = -2e^{-2x}$

$f''(x) = 4e^{-2x}$

$f'''(x) = -8e^{-2x}$

$f^4(x) = 16e^{-2x}$

$R_3(x) = \frac{\max|f^4(z)|}{4!}(x-0)^4 < 0.001$

$= \frac{16e^{-2z}}{4!}x^4 < 0.001$

$= \frac{16e^{-2z}}{24}x^4 < 0.001$

$e^{-2z}x^4 < 0.0015$

$\left(x^4\right)^{1/4} < \left(\frac{0.0015}{e^{-2z}}\right)^{1/4}$

$|x| < 0.1970e^{1/2z} < 0.1970$

$\boxed{0 < x < 0.1970}$