AP Calculus BC 9.9 Notes Geometric Power Series

Key

We need to be able to represent certain types of rational functions as a geometric series. Rather than producing the Taylor's Rule, we will want to develop the series by manipulating a geometric series, or in some cases, using Long Division

Example 1:
First we'll do a quick review of geometric series. Geometric series are formed by multiplying by a common ratio r. Suppose I told you to start with $a_1 = 2$ and to let $r = 3$, what geometric series would you write? WHAT WOULD THE SUM BE IN EACH CASE??

$$\sum_{n=0}^{\infty} 2(3)^n \qquad \sum_{n=0}^{\infty} ar^n = \frac{a_1}{1-r}, \ |r|<1$$

What if $a_1 = 2$ and $r = -3$?

$$\sum_{n=0}^{\infty} 2(-3)^n$$

What if $a_1 = 1$ and $r = x$?

$$\sum_{n=0}^{\infty} 1(x)^n \quad \text{If } |x|<1, \ S = \frac{a_1}{1-r} \rightarrow S = \frac{1}{1-x}$$

Example 2: Find the power series for $\dfrac{1}{1-x}$ centered at c = 0 by

1 ← dividend; $1-x$ ← divisor centered at $c=0$

$\frac{f^{(n)}(c)}{n!}(x-c)^n$

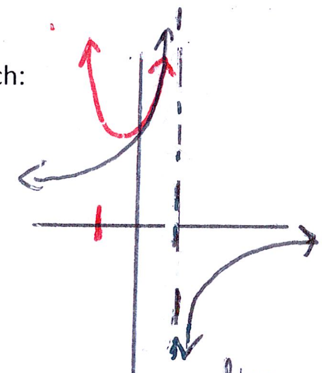
a) Using Taylor's Rule (safety net)
b) Manipulating a geometric series
c) Doing Long Division.

Ex. 2 b) $S = \frac{a_1}{1-r} = \frac{1}{1-x}$

$= 1 + x + x^2 + x^3 + \ldots x^n$

centered at $x=0$ $\quad |r|<1 \quad -1<x<1$

Find the Interval of Convergence (without using the ratio test!) Verify by graphing each:



$$1-x \overline{)\,1} \quad 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{x-1} = \sum_{n=0}^{\infty} x^n$$

$$\begin{array}{l} -1 + x \\ \hline x \\ -x + x^2 \\ \hline x^2 \\ x^2 - x^3 \\ \hline \end{array}$$

radius

$|x-0|<1$

Find I.O.C. $\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x| < 1$

possible convergence $[(-1,1)]$

I.O.C: $(-1,1)$

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Example 3:

I.O.C: $(-1,1)$

Find a power series for $\frac{1}{1+x}$, then find the interval of convergence. Include the first four nonzero terms and the general term.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n \quad \text{or } (-x)^n$$

Example 4:

Find a power series that represents $\frac{x}{1+x}$, then find the interval of convergence. Include the first four nonzero terms and the general term.

$$\frac{x}{1+x} = x\left(1 - x + x^2 - x^3 + \ldots\right)$$

$$= x - x^2 + x^3 - x^4 + \ldots + (-1)^n x^{n+1}$$

Example 5:

centered at $x = 0$

Find a power series for $f(x) = \frac{1}{1-x^2}$, then find the interval of convergence. Find the first four nonzero terms and the general term.

* Think Geometric Series

$$S = \frac{a_1}{1-r}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \ldots + x^{2n} + \ldots$$

Example 6: $r = |2x| < 1 \quad -\tfrac{1}{2} < x < \tfrac{1}{2}$

Find a power series that represents $\frac{1}{1-2x}$, then find the interval of convergence. Include the first four nonzero terms and the general term.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

$$\frac{1}{1-2x} = 1 + (2x) + (2x)^2 + (2x)^3 + \ldots + (2x)^n$$

$$1-2x \overline{\smash{)}\,1\,}\quad 1 + 2x + 4x^2$$

$-(1-2x)$

$2x + 4x^2$

Example 7: *centered at $x = 0$

Find a power series for $g(x) = \frac{1}{4+x}$, centered at then find the interval of convergence. Include the first four nonzero terms and the general term.

*match to $\frac{a_1}{1-r}$

$$\frac{1}{4+x} = \frac{1}{4-(-x)} \rightarrow \frac{\frac{1}{4}}{\frac{4}{4} - \left(\frac{-x}{4}\right)}$$

a_1

ratio

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)\left(\frac{-x}{4}\right)^n = \frac{(-1)^n x^n}{4^{n+1}}$$

$$= \frac{1}{4} - \frac{x}{4^2} + \frac{x^2}{4^3} - \frac{x^3}{4^4} + \ldots \frac{(-1)^n x^n}{4^{n+1}}$$

By GST

$\left|\frac{-x}{4}\right| < 1$

$-4 < x < 4$

center $c = 0$

$r = 4$

I.O.C. $(-4, 4)$

Sometimes we cannot center our function at $x = 0$. In this case, we must try to rewrite our function with the new center showing.

Example 8:

Find a power series that represents $\frac{1}{x}$ centered at $c = 1$, then find the interval of convergence. Include the first four nonzero terms and the general term.

$\boxed{\frac{a_1}{1-r}}$

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \frac{1}{1-(-(x-1))}$$

$f(x) = \frac{1}{x} \quad f(1) = 1$

$f'(x) = \frac{-1}{x^2} \quad f'(1) = -1$

$f''(x) = \frac{2}{x^3} \quad f''(1) = 2$

$f'''(x) = \frac{-6}{x^4} \quad f'''(1) = -6$

$$\frac{1}{x} = 1 - (x-1) + \frac{2}{2!}(x-1)^2 - \frac{6}{3!}(x-1)^3$$

$$1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots (-1)^n (x-1)^n + \ldots$$

Need point that satisfies the graph

$\ln 1 = 0$

$$\int \frac{1}{x}dx = \ln x = x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \ldots \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

$$F(x) = 0 = C + 1 - 0 + 0 - 0 \ldots \quad \boxed{-1 = C}$$

$$\int \frac{1}{x}dx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \ldots \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Example 9:

$\nearrow \dfrac{a_1}{1-r}$

Find a power series for $h(x)=\dfrac{15}{2x-1}$, centered at $c=1$, then find the interval of convergence. Include the first four nonzero terms and the general term.

$$15\left[\frac{1}{2x-1}\right] = 15\left[\frac{1}{2(x-1)+2-1}\right]$$

force the center

$$= 15\left[\frac{1}{2(x-1)+1}\right] = 15\left[\frac{1}{1+2(x-1)}\right]$$

$$15\left[\frac{1}{1-[-2(x-1)]}\right]$$

$$15\left[1-2(x-1)+4(x-1)^2-8(x-1)^3+\dots (-1)^n 2^n (x-1)^n\right]$$

$$= 15-30(x-1)+60(x-1)^2-120(x-1)^3+\dots(-1)2^n\cdot 15(x-1)^n$$

$|r| = |-2(x-1)| < 1$

$|2(x-1)| < 1$

$|x-1| < \frac{1}{2}$

$R = \frac{1}{2}$

I.O.C: $\left(\frac{+1}{2}, \frac{3}{2}\right)$

We can integrate or differentiate a power series to obtain a new series. When we do this, the radius of convergence will be the same, but the interval may change (retest endpoints).

Example 10:

original I.O.C. $|x|<1$ $(-1,1)$

Find a power series that represents $\dfrac{1}{(1-x)^2}$ centered at $c=0$. Hint: what is $\displaystyle\int\left(\frac{1}{(1-x)^2}\right)dx$? What is the radius of convergence?

since $\displaystyle\int \frac{1}{(1-x)^2}dx =$ $u=1-x$, $\frac{du}{dx}=-1$, $dx=-du$ $\displaystyle -\int\frac{1}{u^2}du = -\int u^{-2}du = \frac{-u^{-1}}{-1} = \frac{1}{(1-x)}$

And $\dfrac{1}{1-x} = 1+x+x^2+x^3+x^4+\dots+x^n+\dots$

so $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2} = 0+1+2x+3x^2+4x^3+\dots+nx^{n-1}+\dots$ $\displaystyle\sum_{n=1}^{\infty} n\cdot x^{n-1}$

Retest endpts: at $x=-1$ $\sum n(-1)^{n-1}$ Diverges by nth term

at $x=1$: $\sum n(1)^{n-1} = n$ Diverges by nth term

new I.O.C: $(-1,1)$

* Sometimes a power series is either a derivative or an integral away from another series use-able with known rules

Example 11:

Find a power series that represents $\ln(1-x)$ centered at $c=0$. Hint: what is $\displaystyle\int\left(\frac{1}{1-x}\right)dx$? What is the radius of convergence? $(-1,1)$

$r=x$

$$\frac{1}{1-x} = \frac{1}{1-(x)} = 1+x+x^2+x^3+\dots+x^n$$

Test value at center (guaranteed shared point)

$$\int\frac{1}{1-x}dx = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\dots+\frac{x^n}{n}$$

Since $\dfrac{d}{dx}\ln(1-x) = \dfrac{-1}{1-x}$, $\displaystyle\int\frac{-1}{1-x}dx = \ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\dots-\frac{x^{n+1}}{n+1}+C$

at $x=0$ $\ln(1-x) = C-x-\frac{x^2}{2}-\frac{x^3}{3}-\dots$

$0 = C-0+0-0$ $\boxed{C=0}$

$$\sum_{n=0}^{\infty}\frac{-x^{n+1}}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-x^{n+2}}{n+2}\cdot\frac{n+1}{-x^{n+1}}\right| = |x|<1 \quad [(-1,1)]$$

I.O.C: $[-1,1)$

Example 12:

(Similar to 2008—BC6B) Let f be the function given by $f(x)=\dfrac{1}{1+x^2}$. $= \dfrac{1}{1-(-x^2)}$ ← modified Geometric Series

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x=0$.

(b) Does the series found in part (a), when evaluated at $x=1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\arctan x$ about $x=0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|\underset{\text{approximation}}{A}-\underbrace{\arctan\left(\frac{3}{4}\right)}_{\text{actual}}\right|<\underset{\text{error/remainder}}{\frac{1}{10}}$. Justify your answer.

a) $1-x^2+x^4-x^6+\ldots+(-1)^n(x^2)^n+\ldots$

b) ratio $=|r|=|-x^2|<1 \qquad |x^2|<1 \rightarrow |x|<1$

I.O.C. $(-1,1)$. Since this is a geometric series, no convergence at $x=1$, (endpoint)

c) $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$

$\displaystyle\int \frac{1}{1+x^2}\,dx = \int 1-x^2+x^4-x^6+\ldots(-1)^n x^{2n}\,dx$

$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ ← plug in $x=0$ to find "C"

$\arctan 0 = C+0-0\ldots \qquad C=0$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

d) $\arctan\left(\frac{3}{4}\right) = \frac{3}{4} - \frac{(3/4)^3}{3} + \frac{(3/4)^5}{5} - \frac{(3/4)^7}{7} + \ldots$

$A = \frac{3}{4} - \left(\frac{3}{4}\right)^3\left(\frac{1}{3}\right) = \boxed{\frac{39}{64}}$ since series is a converging alternating series, the error bound is the first unused term: (3rd term $\left(\frac{3}{4}\right)^5\left(\frac{1}{5}\right) < \frac{1}{10}$)

$\frac{3^5}{4^5 \cdot 5} < \frac{1}{10}$