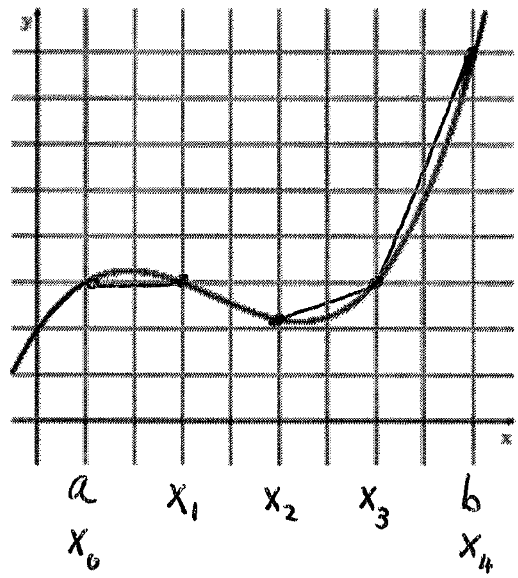
Key

BC Calculus – 8.5 Notes – Arc Length of Curve and Distance Traveled

The idea behind finding arc length is very similar to the way we find area using calculus. We are going to divide the curve into a large quantity of small segments, find their lengths and then add them up. → make the number of line segments approach infinity

Recall the distance formula: $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

We can approximate the length of the graph of a function f by using n line segments whose endpoints are partitioned between $[a,b]$. Think of it this way: $a = x_0$ and $b = x_n$ where $x_0 < x_1 < x_2 < x_3 < \cdots < x_n$.



Arc Length

differentiable curve

If a function $y = f(x)$ represents a smooth continuous curve on the closed interval $[a,b]$, the arc length of f between a and b is given by

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

*we use $f'(x)$ in the formula b/c of the connection to the infinite numbers of tangent line segments we are adding together

1. Find the arc length of the graph $y = \frac{1}{12}(x^2+8)^{\frac{3}{2}}$ from $x = 1$ to $x = 4$.

$y' = \frac{3}{2}\cdot\frac{1}{12}(x^2+8)^{1/2}\cdot(2x)$

$y' = \frac{x}{4}(x^2+8)^{1/2} \rightarrow f'(x)$

$\int_1^4 \sqrt{1+\left[\frac{x}{4}(x^2+8)^{1/2}\right]^2}\,dx$

$\int_1^4 \sqrt{1+\frac{x^2}{16}(x^2+8)}\,dx$

$\int \sqrt{\frac{1}{16}\left[16+x^2(x^2+8)\right]}\,dx$

$\frac{1}{4}\int_1^4 \sqrt{x^4+8x^2+16}$

$\frac{1}{4}\int_1^4 \sqrt{(x^2+4)^2}\,dx$

$\frac{1}{4}\int_1^4 (x^2+4)\,dx = \boxed{\frac{33}{4}}$

Often the problem will only ask that you set up the integral, but we could take this further and use the calculator

2. Set up an integral that represents the length of the curve $y = \sin x$, for $0 \le x \le \pi$ and use a calculator to find the value.

$y' = \cos x$

$$\int_0^{\pi} \sqrt{1 + (\cos x)^2}\, dx \approx \boxed{3.820}$$

Proof:

1. $$L \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

2. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

3. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta x_i}{\Delta x_i}\right)^2 (\Delta y_i)^2}$$

4. $$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2 (\Delta x_i)^2}$$

5. $$L \approx \sum_{i=1}^{n} \sqrt{\left[1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2\right] \cdot (\Delta x_i)^2}$$

6. $$L \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

7. The approximation improves as we take the number of segments to infinity.
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i$$

8. Because $f'(x)$ exists for every x in the interval (x_{i-1}, x_i) the MVT guarantees that there is a value c_i in the interval such that
$$\frac{f(x_i) - f(x_{i-1})}{(x_i - x_{i-1})} = f'(c_i)$$

9. $$f'(c_i) = \frac{\Delta y_i}{\Delta x_i}$$

10. $$L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \cdot \Delta x_i$$

11. Using the definition of integration, we get
$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Practice Problems:

1. Find an expression for the length of the curve $y = \sin x$ from $x = 0$ to $x = \frac{5\pi}{6}$. Do Not Evaluate.

$f'(x) = \cos x$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\boxed{L = \int_0^{5\pi/6} \sqrt{1 + \cos^2 x}\, dx}$$

2. The length of a curve from $x = 1$ to $x = 3$ is given by $\int_1^3 \sqrt{1+4x^2}\, dx$. If the point $(1, 6)$ is on the curve, which of the following could be an equation for this curve?

A. $y = \frac{4}{3}x^3 + x + 1$

B. $y = 4x^2 + 1$

C. $\boxed{y = x^2 + 5}$

D. $y = x^2 - 6$

E. $1 + \frac{4}{3}x^3$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$[f'(x)]^2 = 4x^2$$

$$f'(x) = 2x$$

$$f(x) = \frac{2x^2}{2} + C$$

$y = x^2 + C$ ← point $(1, 6)$ x y

$$6 = 1^2 + C$$

$$5 = C$$

$$\boxed{y = x^2 + 5}$$

3. **Calculator active.** Suppose $G(x) = \int_0^x \sqrt{\sin(t)}\, dt$, for $0 \le x \le \pi$. What is the length of the arc along the curve $y = G(x)$ for $x = 0$ to $x = \pi/7$.

$$G'(x) = \frac{d}{dx}\int_0^x \sqrt{\sin(t)} = \sqrt{\sin x}$$

$$L = \int_0^{\pi/7} \sqrt{1 + [\sqrt{\sin x}]^2}\, dx \approx \boxed{0.495}$$

4. **No Calculator.** Let $g(x) = \sqrt{3x}$ and f be an antiderivative of g. $f(x) = \int \sqrt{3x}\, dx$

a. Find $f'(x)$

$$f'(x) = \frac{d}{dx}\int \sqrt{3x}\, dx = \sqrt{3x}$$

b. Find an expression for the length of the graph of f from $x = a$ to $x = b$.

$$L = \int_a^b \sqrt{1 + [\sqrt{3x}]^2}\, dx = \int_a^b \sqrt{1 + 3x}\, dx$$

c. If $a = 0$ and $b = 8$, find the length of the graph of f from a to b.

$$\int_0^8 \sqrt{1 + 3x}\, dx$$

$u = 1 + 3x$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$$\int u^{1/2} \cdot \frac{du}{3}$$

$$\frac{1}{3}\int u^{1/2}\, du$$

$$\frac{1}{3} \cdot \frac{u^{3/2}}{3/2}$$

$$\frac{1}{3} \cdot \frac{2}{3}(1 + 3x)^{3/2}\Big]_0^8$$

$$\frac{2}{9}(25)^{3/2} - \frac{2}{9}(1)^{3/2}$$

$$\frac{2}{9}(5)^3 - \frac{2}{9}(1)$$

$$\frac{2}{9}(5^3 - 1^3)$$

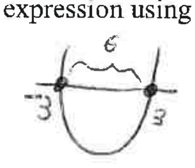
$$\frac{2}{9}(124) = \boxed{\frac{248}{9}}$$

5. **Calculator active.** Consider the region bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = 5$.

a. Write an expression using one or more integrals that could be used to find the perimeter of this region.

$$x^2 - 4 = 5$$

$$x = \pm 3$$



$$\int_{-3}^{3} \sqrt{1 + (2x)^2}\, dx + \int_{-3}^{3} \sqrt{1 + (0)^2}\, dx$$

(curve) + (line)

b. Find the perimeter.

$$19.494 + 6 = 25.494$$

$$\boxed{25.494}$$

6. Find an integral that gives the length of the graph $y = \cos\sqrt{x}$ between $x = a$ and $x = b$, where $0 < a < b$.

$y' = -\sin\sqrt{x} \cdot \frac{1}{2}x^{-1/2} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$

$\int_a^b \sqrt{1 + \left[\frac{1}{2\sqrt{x}}\sin\sqrt{x}\right]^2}\,dx \rightarrow \boxed{\int_a^b \sqrt{1 + \frac{1}{4x}\sin^2\sqrt{x}}\,dx}$

7. **Calculator active.** Let f be a function with derivative $f'(x) = \sqrt{x^5 + 1}$. What is the length of the graph of $y = f(x)$ from $x = 0$ to $x = 2.5$?

$L = \int_0^{2.5} \sqrt{1 + (\sqrt{x^5+1})^2}\,dx \approx \boxed{8.688}$

$f(x) = \frac{5}{7}x^3 - \frac{2}{7}x - \frac{1}{7}$

8. Find an integral that is equal to the length of the curve $f(x) = \frac{5x^3 - 2x - 1}{7}$ from the point $(0, -0.143)$ to the point $(2, 5)$.

$f'(x) = \frac{5}{7}\cdot 3x^2 - \frac{2}{7} = \frac{15x^2}{7} - \frac{2}{7}$

$\boxed{L = \int_0^2 \sqrt{1 + \left(\frac{15x^2 - 2}{7}\right)^2}\,dx}$

9. Find an expression for the length of the graph of $y = e^{3x}$ between $x = 1$ and $x = 3$.

$y' = e^{3x} \cdot 3$

$L = \int_1^3 \sqrt{1 + (3e^{3x})^2}\,dx = \boxed{\int_1^3 \sqrt{1 + 9e^{6x}}\,dx}$

10. **Calculator active.** The trajectory of a ball thrown from a height of 160 meters is given by the equation $y = 160 - \frac{x^2}{40}$ until it hits the water where y is the height of the ball above the water and x is the horizontal distance traveled in meters. Find the distance traveled by the ball from the time it is thrown until it hits the water.

height of water

$0 = 160 - \frac{x^2}{40}$

$\frac{x^2}{40} = 160$

$x^2 = 6400 \rightarrow x = 80$

$f'(x) = \frac{-1}{40} \cdot 2x$

$\boxed{L = \int_0^{80} \sqrt{1 + \left(\frac{-x}{20}\right)^2}\,dx \approx 185.871}$

Arc Length

11. Which of the following integrals gives the length of the curve $y = \frac{1}{2}x^3$ from $x = 1$ to $x = 3$?

$y' = \frac{3}{2}x^2$

$L = \int_1^3 \sqrt{1 + \left(\frac{3}{2}x^2\right)^2}\,dx \rightarrow \sqrt{1 + \frac{9}{4}x^4} \rightarrow \sqrt{\frac{4}{4} + \frac{9x^4}{4}} \rightarrow \sqrt{\frac{4 + 9x^4}{4}}$

$\sqrt{\frac{1}{4}(4 + 9x^4)} \longrightarrow \boxed{\frac{1}{2}\sqrt{4 + 9x^4}}$

A. $\int_1^3 \sqrt{1 + \frac{1}{4}x^6}\,dx$

B. $\int_1^3 \sqrt{1 + \frac{1}{2}x^6}\,dx$

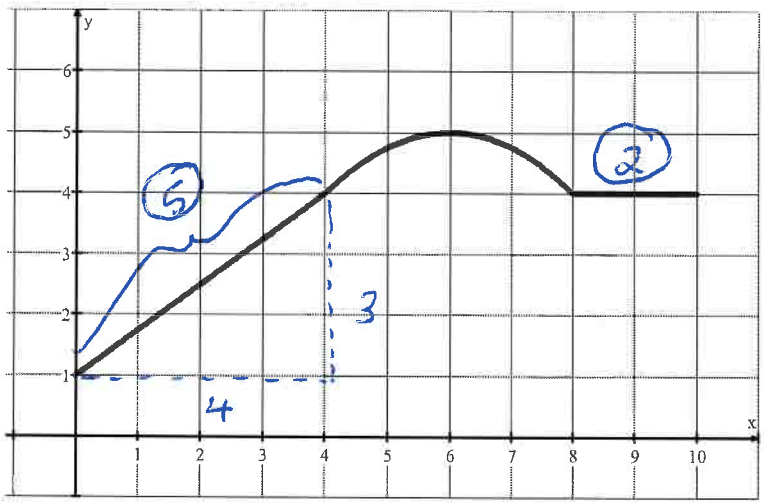
$\boxed{\text{C.}}$ $\int_1^3 \frac{1}{2}\sqrt{4 + 9x^4}\,dx$

D. $\int_1^3 \sqrt{1 + \frac{3}{2}x^4}\,dx$

12. **Calculator active**. What is the length of the curve $y = 1 - \sin x$ from $x = 0$ to $x = 4\pi$?

$y' = -\cos x$ $\qquad L = \int_0^{4\pi} \sqrt{1 + [\cos x]^2}\, dx \approx \boxed{15.2807}$

13.



$$f(x) = \begin{cases} 1 + \dfrac{3}{4}x & \text{for } 0 \le x < 4 \\ 5 - \dfrac{1}{4}(x-6)^2 & \text{for } 4 \le x < 8 \\ 4 & \text{for } 8 \le x \le 10 \end{cases}$$

A mountain hike consists of a steady incline followed by a curved hill and then a flat valley. The mountain hike is modeled by the piecewise-defined function f above, and the graph of f is shown in the figure above. Which of the following expressions gives the total length of the hike from $x = 0$ to $x = 10$.

$f'(x) = \frac{-1}{4} \cdot 2(x-6) = \frac{-1}{2}(x-6) = \frac{-x}{2} + 3$

$\hookrightarrow [f'(x)]^2 = \left[\frac{-1}{2}(x-6)\right]^2 \to \frac{1}{4}(x-6)^2 \to \int_4^8 \sqrt{1 + \frac{1}{4}(x-6)^2}\, dx$

A. $2 + \int_0^8 \sqrt{1 + \left(\frac{3}{4} - \frac{1}{2}(x-6)\right)^2}\, dx$

B. $2 + \int_0^8 \sqrt{1 + \left(\frac{3}{4}\right)^2} + \sqrt{1 - \frac{1}{4}(x-6)^2}\, dx$

C. $7 + \int_4^8 \sqrt{1 + \left(1 - \frac{1}{2}(x-6)^2\right)^2}\, dx$

D. $\boxed{7 + \int_4^8 \sqrt{1 + \frac{1}{4}(x-6)^2}\, dx}$ $\leftarrow$