Key

10.10 AP Practice Problems (p.806) – Taylor Polynomial Approximations & Lagrange Error Bound

1. Use the Taylor polynomial $P_4(x)$ of $y = \cos x$ at 0 to approximate $\cos 0.2$.

(A) 1 (B) 0.980 (C) 0.803 (D) 0.801

$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$P_4(0.2) = 1 - \frac{0.2^2}{2!} + \frac{0.2^4}{4!}$

$\boxed{P_4(0.2) = 0.980}$

2. If a function f can be represented by the Taylor polynomial

$$P_3(x) = 4 + 2(x-2) + 3(x-2)^2 + \frac{1}{2}(x-2)^3$$

then $f''(2) =$

(A) 1 (B) 2 (C) 3 (D) 6

$f'(x) = 0 + 2 + 6(x-2) + \frac{3}{2}(x-2)^2$

$f''(x) = 0 + 6 + \frac{3}{2}\cdot 2(x-2)$

$f''(2) = 6 + \frac{3}{2}\cdot 2(2-2) = \boxed{6}$

3. If $f(-1) = 4$, $f'(-1) = -3$, $f''(-1) = 3$, and $f'''(-1) = 2$ then the Taylor polynomial $P_3(x)$ of degree 3 of f at -1 is

(A) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{2}{3}(x+1)^3$

(B) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 - \frac{1}{3}(x+1)^3$

(C) $P_3(x) = 4 - 3(x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

(D) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{1}{3}(x+1)^3$

Taylor polynomial $\sum_{n=0} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$

$P_3(-1) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 + \ldots$

$= \boxed{4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{1}{3}(x+1)^3}$

4. The polynomial $f(x) = x^3 - x^2 - 4$ expressed in powers of $(x+1)$ equals

(A) $-4 + (x+1) + 4(x+1)^2 + 6(x+1)^3$

(B) $-6 + 5(x+1) + 4(x+1)^2 + (x+1)^3$

(C) $-6 + 5(x+1) - 8(x+1)^2 + 6(x+1)^3$

(D) $-6 + 5(x+1) - 4(x+1)^2 + (x+1)^3$

$f(-1) = -6$

$f'(x) = 3x^2 - 2x$

$f'(-1) = 3 + 2 = 5$

$f''(x) = 6x - 2$

$f''(-1) = -8$

$f'''(x) = 6 \quad f'''(-1) = 6$

$$P_3(x) = f(-1) + f'(-1)(x-c) + \frac{f''(-1)}{2!}(x-c)^2 + \frac{f'''(-1)}{3!}(x-c)^3$$

$$P_3(x) = -6 + 5(x+1) - \frac{8}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$

$$\boxed{P_3(x) = -6 + 5(x+1) - 4(x+1)^2 + (x+1)^3}$$

5. The Lagrange error bound in using a Taylor polynomial of degree 5 for $f(x) = \sin x$ at $\frac{2\pi}{3}$ to approximate $\sin 2$ is no more than

(A) $\frac{\left|\frac{2\pi}{3} - 2\right|^5}{5!}$ (B) $\frac{\left|\frac{2\pi}{3} - 2\right|^6}{6!}$

(C) $\frac{\left|\cos\frac{2\pi}{3}\right|\left|\frac{2\pi}{3} - 2\right|^6}{6!}$ (D) $\frac{\left|\sin\frac{2\pi}{3}\right|\left|\frac{2\pi}{3} - 2\right|^6}{6!}$

$f(x) = \sin x$

$f^6(x) \leq 1$

$c = 2$

$x = 2\pi/3$

$$*\text{Lagrange}: |R_n(x)| \leq \left|\frac{\text{Max}\left[f^{n+1}(z)\right](x-c)^{n+1}}{(n+1)!}\right|$$

$$|R_5(2)| \leq \left|\frac{(1)\left(\frac{2\pi}{3} - 2\right)^{5+1}}{(5+1)!}\right|$$

$$\leq \boxed{\left|\frac{(2\pi/3 - 2)^6}{6!}\right|}$$

6. (a) Write the Maclaurin series for $f(x) = e^{-x}$.
(b) Approximate $e^{-0.1}$ with the Taylor polynomial $P_2(x)$ of $f(x) = e^{-x}$.
(c) How many terms are necessary to approximate $e^{-0.1}$ with an error less than or equal to 0.00001?

a) $\cdots \frac{(-x)^n}{n!}$

b) 0.905

c) 4 terms

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$

$e^{-x} = 1 + (-x) + \frac{x^2}{2!} + \frac{-x^3}{3!} + \ldots + \frac{(-x)^n}{n!} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

b) $e^{-0.1} \approx 1 + (-0.1) + \frac{(-0.1)^2}{2} = \boxed{0.905}$

c) 4^{th} term $(n=3) \rightarrow \left|\frac{(-1)^3 (0.1)^3}{3!}\right| = 0.00016$

5^{th} term $(n=4) \rightarrow \left|\frac{(-1)^4 (0.1)^4}{4!}\right| = 0.00000416 < 0.00001$

Since the value of the 5^{th} term meets the desired approximate error, $\boxed{\text{4 terms are sufficient.}}$

7. The standard normal probability distribution density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is defined for all real numbers.

(a) Write the first four nonzero terms of the Maclaurin expansion for f.

(b) Use the Maclaurin expansion from (a) and properties of a power series to approximate $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$.

(c) Find the maximum error in using the first four terms of the Maclaurin expansion to approximate $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$.

OMIT (d) Justify the method used to find the error in (c).

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^{-x^2/2} = 1 + \frac{-x^2}{2} + \frac{(-x^2/2)^2}{2!} + \frac{(-x^2/2)^3}{3!} \rightarrow 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$

$\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \boxed{\frac{1}{\sqrt{2\pi}}\left[1 - \frac{x^2}{2} + \frac{1}{8}x^4 - \frac{1}{48}x^6\right]}$

b) $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \rightarrow \frac{1}{\sqrt{2\pi}}\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336}\right]_0^1 = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336}\right)$

$= \frac{1}{\sqrt{2\pi}}\left(\frac{479}{560}\right) \approx \boxed{0.341}$

c) $|E_4| \leq a_5 \rightarrow \frac{(-x^2/2)^4}{4!}$ ← 5th term

$\frac{x^8}{16 \cdot 4!} \rightarrow \frac{x^8}{384} \rightarrow \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{384}\int_0^1 x^8 \rightarrow \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{384} \cdot \frac{x^9}{9}\Big]_0^1$

$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{384} \cdot \frac{1}{9} \rightarrow \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{3456}$

$\boxed{\approx 1.154 \times 10^{-4}}$