Key

BC Calculus – 10.10b Notes – Lagrange Error Bound

Error bound means how far off is the approximation from the actual value or answer

Exact value = Approximate value + Remainder

$$f(x) = \underset{\text{(Taylor polynomial)}}{P(x)} + \underset{\text{(Remainder)}}{R(x)}$$

$$R(x) = f(x) - P(x)$$

Error: $|R_n(x)| = |f(x) - P_n(x)|$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!} + \boxed{R(x)}$$

$$\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!} \rightarrow \frac{\boxed{f^{(n+1)}(z)}(x-c)^{n+1}}{(n+1)!}$$

*we want to know what is the greatest possible value of this (n+1) derivative on an interval

Lagrange Error Bound

Let $f(x)$ be differentiable through the order $n+1$. The error between the Taylor Polynomial and $f(x)$ is bounded by:

$$|R_n(x)| \le \left| \frac{\text{Max}\left[f^{(n+1)}(z)\right](x-c)^{n+1}}{(n+1)!} \right|$$

where z is some number between c and x.

1. **The fourth degree Maclaurin polynomial (centered at $x=0$) for $\cos x$ is given by $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. If this polynomial is used to approximate $\cos(0.2)$, what is the Lagrange error bound?**

$$\cos(0.2) \approx P_4(0.2) = 0.980067$$

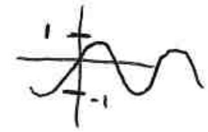
$x = 0.2$
$c = 0$
$n = 4$

$$R_4(x) \le \left| \frac{\text{Max}(-\sin(z))(0.2-0)^5}{(4+1)^5} \right|$$

$* \quad c \le z \le x$

$0 \le z \le 0.2$

sinx curve



$f = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f'''(x) = \sin x$
$f^4(x) = \cos x$
$f^5(x) = -\sin x$

$$R_4(x) \le \left| \frac{1 \cdot (0.2)^5}{5!} \right|$$

$$R_4(x) \le 2.667 \times 10^{-6}$$

$$R_4(x) \le 0.000002667$$

2. Use a third degree Taylor polynomial on the interval $[0, 1]$ for e^x centered about $x = 0$ to approximate e^1. What is the error bound of this approximation?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^1 \approx P_3(1) = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} = 2.66667$$

$$R_3(x) \leq \left| \frac{\max[e^z] \cdot (1-0)^4}{4!} \right| \qquad 0 \leq z \leq 1$$

$$R_3(x) \leq \left| \frac{e^1 (1)^4}{4!} \right|$$

$$\boxed{R_3 \leq 0.1132617}$$

3. What is the smallest order Taylor Polynomial centered at $x = 1$ which will approximate e^{x-1} on the interval $[0, 3]$ with a Lagrange error bound less than 1?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots$$

* $0 < z < 3$

$$\left| \frac{\text{Max}\left(f^{(n+1)}(z)\right) \cdot (x-1)^{n+1}}{(n+1)!} \right| < 1$$

$x = 3$

$$\left| \frac{e^2 \cdot (3-1)^{n+1}}{(n+1)!} \right| < 1$$

$$\frac{e^2 \cdot 2^{n+1}}{(n+1)!} < 1$$

n	R
1	14.778
2	9.8521
3	4.926
4	1.9704
5	0.6568

$\boxed{\text{5th order Taylor Polynomial}}$

Practice Problems:

1. The third Maclaurin polynomial for $\sin x$ is given by $p(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.1)$, what is the Lagrange error bound?

Interval $[\overset{c}{0}, \overset{x}{0.1}]$

$$\frac{\max\left[f^{n+1}(z)\right]\left[x-c\right]^{n+1}}{(n+1)!}$$

$$\frac{\text{Max}\left[f^4(z)\right]\left[0.1-0\right]^4}{4!}$$

$f = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f^3(x) = -\cos x$
$f^4(x) = \sin x$

max of $\sin x$ is $\boxed{1}$

$$\frac{[1][0.1]^4}{4!} \to \boxed{4.1667 \times 10^{-6}}$$

2. If the Taylor Polynomial for approximating $\cos x$ is given by $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, what is the upper bound for the error in the approximation of $\cos(0.3)$?

Interval $[\overset{c}{0}, \overset{x}{0.3}]$

$n = 4$

$$\frac{\text{Max}[f^5(z)][0.3-0]^5}{5!}$$

$f = \cos x$
$f' = -\sin x$
$f'' = -\cos x$
$f''' = \sin x$
$f^4 = \cos x$
$f^5 = -\sin x$
Max of $-\sin x = 1$

$$\frac{(1)(0.3)^5}{5!} = \boxed{2.025 \times 10^{-5}}$$

3. If the Taylor Polynomial about $x = 0$ for the approximation of e^x is given by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$, what is the upper bound for the error in the approximation of e? → e means e^1 → interval $[\overset{c}{0}, \overset{x}{1}]$

$f^6(x) = e^x$

Max on interval $[0,1]$ is $\boxed{e}$

$$\frac{\text{Max}[f^6(z)][1-0]^6}{6!} \rightarrow \frac{(e)(1)^6}{6!} = \boxed{0.00377}$$

4. Let f be a function that has derivatives of all orders for all real numbers and let $P_3(x)$ be the third-degree Taylor Polynomial for f about $x = 0$. $|f^{(n)}(x)| \le \frac{n}{n+1}$, for $1 \le n \le 5$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \le k$?

$R_3(x) \le K$

$\text{Max}[f^4(z)] \le \frac{4}{4+1} = \frac{4}{5}$ on interval $[\overset{c}{0}, \overset{x}{1}]$

$$\frac{\text{Max}[f^4(z)][x-c]^4}{4!} \rightarrow \frac{(\frac{4}{5})(1-0)^4}{4!} \rightarrow \boxed{\frac{4}{5} \cdot \frac{1}{4!}}$$

(A) $\frac{5}{6}$ (B) $\frac{5}{6} * \frac{1}{5!}$ (C) $\frac{5}{6} * \frac{1}{4!}$ (D) $\boxed{\frac{4}{5} * \frac{1}{4!}}$

5. The function f has derivatives of all orders for all real numbers, $f^{(4)}(x) = e^{\cos x}$. If the third-degree Taylor Polynomial for f about $x = 0$ is used to approximate f on the interval $[0,1]$, what is the Lagrange error bound?

$$\frac{\text{Max}[f^4(z)][1-0]^4}{4!}$$

$$\frac{(e^1)(1)^4}{4!} \approx \boxed{0.11326}$$

Max of $e^{\cos x}$ on $[0,1]$ is $x = 0 \rightarrow e^{\cos 0} \rightarrow e^1$

6. The Taylor series for a function f about $x = 3$ is given by $\sum_{n=0}^{\infty}(-1)^n \frac{3n+1}{2^n}(x-3)^n$ and converges to f for $0 \leq x \leq 5$. If the third-degree Taylor Polynomial for f about $x = 3$ is used to approximate $f\left(\frac{13}{4}\right)$, what is the alternating series error bound?

$$a_4 = \frac{3(4)+1}{2^4}(x-3)^4 \rightarrow \frac{13}{16}\left[\frac{13}{4}-3\right]^4$$

$$\boxed{|R_3| \leq \frac{13}{16}\left(\frac{1}{4}\right)^4}$$

7. Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. $T_4(x)$ is the fourth-degree Taylor Polynomial for f about $x = c$ such that $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$. Based on the Lagrange error bound, $f(x) - T_4(x)$ must equal which of the following?

$|R_4(x)| = |f(x) - T_4(x)|$ is the next term

(A) x (B) $(x-c)^5$ (C) $\boxed{a_5(x-c)^5}$ (D) $\frac{a_5(x-c)^5}{5!}$

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8. Let $P(x)$ be the sixth-degree Taylor Polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \leq x \leq 1.5$ is given below.

$\max_{0\leq x\leq 1.5}|f^{(5)}(x)| = 9.3$ $\max_{0\leq x\leq 1.5}|f^{(6)}(x)| = 62.1$ $\max_{0\leq x\leq 1.5}|f^{(7)}(x)| = 481.3$

What is the smallest value of k for which the Lagrange error bound guarantees that $|f(1.5) - P(1.5)| \leq k$?

$$\frac{\max[f^7(z)][x-c]^7}{7!} \longrightarrow \frac{(481.3)(1.5-0)^7}{7!} \approx \boxed{1.6316}$$

on interval $[\overset{c}{0}, \overset{x}{1.5}]$

9. The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 2$ are given in the table.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
2	6	-12	18	-24	34

a. Write the third-degree Taylor Polynomial for f about $x = 2$, and use it to approximate $f(1.5)$.

$$P_3(x) = f(x) + f'(x)[x-2] + \frac{f''(x)}{2!}(x-2)^2 + \frac{f'''(x)}{3!}(x-2)^3$$

$$P_3(x) = 6 + -12(x-2) + \frac{18}{2}(x-2)^2 - \frac{24}{3!}(x-2)^3$$

$$P_3(x) = 6 - 12(x-2) + 9(x-2)^2 - 4(x-2)^3$$

$$f(1.5) \approx P_3(1.5) = 6 - 12(-0.5) + 9(-0.5)^2 - 4(-0.5)^3 = \boxed{14.75}$$

b. The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 48$ for all $x > 1$. Use the Lagrange error bound to show that the approximation found in part (a) differs from $f(1.5)$ by no more than $\frac{1}{8}$.

$$R_3(1.5) \le \frac{\max[f^{4}(z)](1.5-2)^4}{4!} = \frac{48(0.5)^4}{4!} = 0.125$$

$$\boxed{R_3(1.5) \le 0.125 = 1/8}$$

10. Let h be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of h are given for $x = 3$. Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 3$ approximates $h(2.9)$ with an error less than 3×10^{-4}.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	$\frac{3483}{8}$	$\frac{1125}{16}$

interval $[2.9, 3]$
x c

$$R_3(x) \le \frac{\max[h^4(z)](x-c)^4}{4!}$$

$$\le \frac{\left(\frac{1125}{16}\right)(2.9-3)^4}{4!}$$

$$\le \boxed{2.9297 \times 10^{-4}}$$

10.12 Lagrange Error Bound

Test Prep

11. **Calculator allowed.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 3$ are given in the table.

a. Write an equation for the line tangent to the graph of f at $x = 3$ and use it to approximate $f(2.5)$.

point: $(3, 4)$

slope: $f'(3) = -8$

$y - 4 = -8(x-3)$

$y = -8(x-3) + 4$

$y(2.5) = -8(2.5) + 28 = 8$

$\boxed{f(2.5) \approx 8}$

$c=3$

b. Write the third-degree Taylor polynomial for f about $x = 3$, and use it to approximate $f(2.5)$.

$$P_3(x) = f(x) + f'(x)(x-3) + \frac{f''(x)}{2!}(x-3)^2 + \frac{f'''(x)}{3!}(x-3)^3$$

$$P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$$

$$P_3(x) = 4 - 8(x-3) + \frac{14}{2}(x-3)^2 - \frac{22}{3!}(x-3)^3$$

$$f(2.5) \approx P_3(2.5) = 4 - 8(-0.5) + 7(-0.5)^2 - \frac{11}{3}(-0.5)^3 = \boxed{10.208}$$

c. Is there enough information to determine whether f has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

There is not enough information. We don't know if $f'(2.5)=0$ or if $f'(2.5)$ does not exist.
The Taylor Polynomial only gives us an approximation of $f(x)$.

d. The fourth derivative of f satisfies the inequality $|f^{(4)}(x)| \le 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

$$R_3(x) \le \frac{\max[f^4(z)]\,[x-c]^4}{4!}$$

interval $[\overset{x}{2.5}, \overset{c}{3}]$

$$\le \frac{(48)(2.5-3)^4}{4!}$$

$$\le \boxed{0.125}$$

e. What is the coefficient of the $(x-3)^3$ term in the Taylor series for f', the derivative of f, about $x = 3$?

$$f(x) = \ldots\ldots \frac{f^4(x)}{4!}(x-3)^4$$

$$= \frac{30}{4!}(x-3)^4$$

$$f'(x) = \ldots\ldots \frac{4\cdot 30}{4!}(x-3)^3$$

Coefficient is 5

$$\frac{30}{3!}(x-3)^3 \text{ or } \frac{30}{6}(x-3)^3 \rightarrow \boxed{5}(x-3)^3$$