**Name:** ______________________________ **Period:** __________

# BC Calculus

## Unit 10 Packet
## Infinite Series
## (Part 2)

## Taylor Series & Power Series

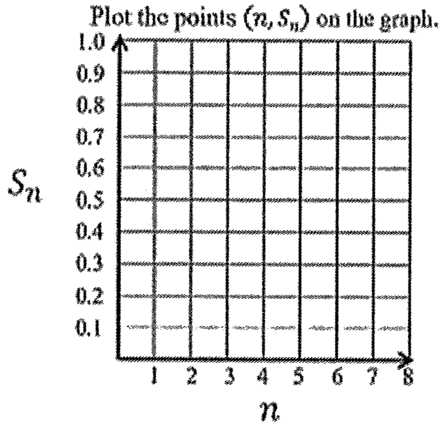
**(Alternating Series Error Bound, Taylor & Maclaurin polynomial and series, Power Series, & Lagrange error bound)**

# BC Calculus – 10.5b Notes –Alternating Series Error Bound

Use the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to fill in the table below.

| $n$ | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | fractions | | | | | | | | |
| | decimals | | | | | | | | |
| $S_n$ | fractions | | | | | | | | |
| | decimals | | | | | | | | |

Plot the points $(n, S_n)$ on the graph.




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Error:** $\lvert S - S_n \rvert$ | | | | | | | | |

**Alternating Series Error Bound**

If you have an alternating series that converges, we can approximate the sum of the series!

$= \quad \leq$

$S$: Sum of the series
$S_n$: Partial sum
$R_n$: Remainder (or error)
$R_n = S - S_n$
$a_{n+1}$ = next term (Error Bound)

(2)

1. Determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ with an error less than $10^{-3}$.

2. If the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n+2}$ is approximated by the partial sum with 10 terms, what is the alternating series error bound?

3. **Calculator active.** Approximate an interval of the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n^2}$ using the Alternating Series Error Bound for the first 5 terms.

4. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$. Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with an error less than 0.01.

5. If the infinite series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n}$ is approximated by $P_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{4}{n}$, what is the least value of $k$ for which the alternating series error bound guarantees that $|S - P_k| < \frac{7}{100}$?

(A) 55 (B) 56 (C) 57 (D) 60

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6. If the series $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$ is approximated by the partial sum $S_k = \sum_{n=1}^{k} (-1)^{n+1} \frac{1}{n^3}$, what is the least value of $k$ for which the alternating series error bound guarantees that $|S - S_k| \leq \frac{7}{10000}$ ?

(A) 10 (B) 11 (C) 12 (D) 13

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7. The series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges by the alternating series test. If $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ is the $n$th partial sum of the series, which of the following statements must be true?

(A) $\lim_{n \to \infty} S_n = 0$ (B) $\lim_{n \to \infty} a_n = S$ (C) $|S - S_{20}| \leq a_{26}$ (D) $|S - S_{25}| \leq a_{26}$

8. If the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{5n+1}$ is approximated by the partials sum with 15 terms, what is the alternating series error bound?

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{76}$ (D) $\frac{1}{81}$

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9. The function $f$ is defined by the power series $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x}{(2n+1)!}$ for all real numbers $x$. Show that $1-\frac{1}{3!}+\frac{1}{5!}$ approximates $f(1)$ with an error less than $\frac{1}{4000}$.

## Alternating Series Error Bound

**Test Prep**

10. **Calculator active!** Let $f(x)=\sum_{n=1}^{\infty}\frac{x^n n^n}{n!}$ for all $x$ for which the series converges.

a. Use the first three terms of the series to approximate $f\left(-\frac{1}{3}\right)$.

b. How far off is this estimate from the value of $f\left(-\frac{1}{3}\right)$? Justify your answer.

11. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is approximated with the series $\sum_{n=1}^{7}(-1)^{n+1}\frac{1}{n^2}$, what is the error bound?

## 10.5 AP Practice Problems (p.765) – Alternating Series Test & Absolute Convergence

1. Which of the following series converge?

I. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$

II. $\sum_{k=1}^{\infty} (-1)^k \left(\frac{5}{3}\right)^k$

III. $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

2. The alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{10^k}$ converges. What is the maximum error incurred by using the first three nonzero terms to approximate the sum of the series?

(A) $-0.083$ (B) 0.003 (C) 0.0004 (D) 0.0826

3. What is the fewest number of terms of the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ that must be added to approximate the sum so that the error is less than or equal to 0.001?

(A) 7 (B) 9 (C) 10 (D) 11

4. Which of the following series converge conditionally, but not absolutely?

I. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{k}$

II. $\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{k}\right)^{4/3}$

III. $\sum_{k=0}^{\infty} (-1)^{k} \left(\frac{3}{4}\right)^{k}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

5. (a) Write out the first five terms of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$.

(b) Show the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ converges.

58 (c) How many terms of the series are necessary to approximate the sum $S$ with an error less than or equal to 0.0001?

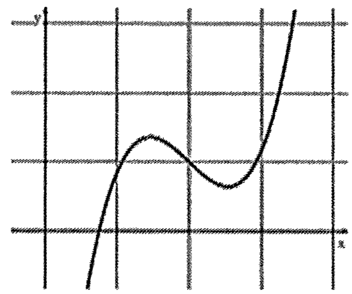
6. Determine whether the series $\sum_{k=1}^{\infty} \frac{\cos(2k)}{4^k}$ converges absolutely, converges conditionally, or diverges. Show your work.

# BC Calculus – 10.10a Notes - Finding Taylor Polynomial Approximations of Functions

**Taylor polynomials** are created to help us approximate other functions. Why would we do this? Because polynomials are easy to work with in calculus (i.e., taking a derivative or integral).

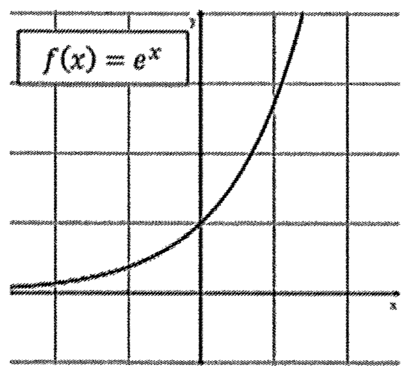
A **Maclaurin polynomial** is a special type of Taylor polynomial.

To start, we choose an $x$-value to center our polynomial approximation. Let's call that $x = c$. Our approximation will have the same $y$-value as the original function at $x = c$.



We **expand the approximation about** $x = c$. Another way of saying this: "the functions are **centered at** $x = c$."

Explore with an example: $f(x) = e^x$. Let $c = 0$.




We want to make the graphs have a similar shape at the point $x = c$. They should have the same slope. That leads us to

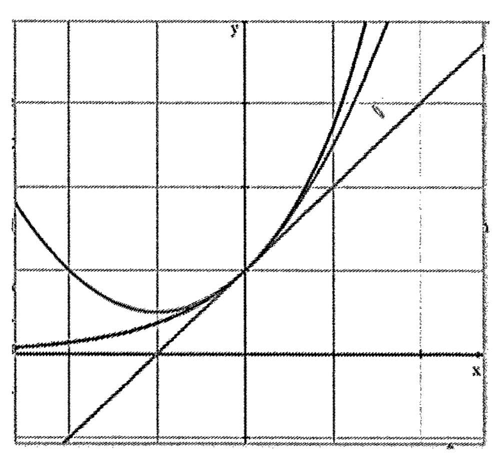
In this example, that means

The polynomial approximation will look like this:

Or rewritten:

This is called a approximation. It works for a small interval around our point of center.

To improve the approximation, make the second derivatives agree at $x = c$.



We want $f(c) = p(c), f'(c) = p'(c), f''(c) = p''(c)$.
For our example this is $f(0) = p(0), f'(0) = p'(0), f''(0) = p''(0)$

If we worked through a similar process, we'd end up with the following:

Second-order approximation:

8

If we are centered at $x = 0$, then we have the following pattern:

$$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

**N[th] Taylor Polynomial**

If $f(x)$ is a differentiable function, then an approximation of $f$ centered about $x = c$ can be modeled by

$p_n(x) =$

where $n$ is the order of the approximation.

**Maclaurin Polynomial**

A Maclaurin polynomial is a Taylor polynomial centered about $x = 0$. It can be modeled by

$p_n(x) =$

where $n$ is the order of the approximation.

1. Find the third-degree Maclaurin polynomial for $f(x) = e^{2x}$

Evaluate at $f(0.2)$ and $p_3(0.2)$

2. Find a fourth-degree Taylor Polynomial for $f(x) = \ln x$ centered at $x = 1$.

Evaluate at $f(1.1)$ and $p_4(1.1)$

## Coefficients of a Taylor Polynomial

The coefficient of the $n$th degree term in a Taylor polynomial for a function $f$ cenetered at $x = c$ is

3. Let $f$ be a function with third derivative $f'''(x) = (8x + 2)^{\frac{3}{2}}$. What is the coefficient of $(x - 2)^4$ in the fourth-degree Taylor Polynomial for $f$ about $x = 2$.

**Practice Problems**

1. Find the fourth-degree Maclaurin Polynomial for $e^{4x}$.

2. Find the fifth-degree Maclaurin Polynomial for the function $f(x) = \sin x$.

(10)

3. Find the third-degree Taylor Polynomial for $f(x) = \ln(2x)$ about $x = 1$.

4. Find the third-degree Taylor Polynomial about $x = 0$ for $\ln(1 - x)$.

5. Find the third-degree Taylor Polynomial about $x = 4$ of $\sqrt{x}$.

6. The function $f$ has derivatives of all orders for all real numbers with $f(1) = -1$, $f'(1) = 4$, $f''(1) = 6$, and $f'''(1) = 12$. Using the third-degree polynomial for $f$ about $x = 1$, what is the approximation of $f(1.1)$?

7. A function $f$ has a Maclaurin series given by $3 + 4x + 2x^2 + \frac{1}{3}x^3 + \cdots$, and the series converges to $f(x)$ for all real numbers $x$. If $g$ is the function defined by $g(x) = e^{f(x)}$, what is the coefficient of $x$ in the Maclaurin series for $g$?

8. Let $f$ be a function with third derivative $f'''(x) = (7x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for $f$ about $x = 2$?

9. Let $P(x) = 4x^2 - 6x^3 + 8x^4 + 4x^5$ be the fifth-degree Taylor Polynomial for the function $f$ about $x = 0$. What is the value of $f'''(0)$?

10. Let $P$ be the second-degree Taylor Polynomial for $f(x) = e^{-3x}$ about $x = 3$. What is the slope of the line tangent to the graph of $P$ at $x = 3$?

11. Let $f$ be a function with $f(4) = 2$, $f'(4) = -1$, $f''(4) = 6$, and $f'''(4) = 12$. What is the third-degree Taylor Polynomial for $f$ about $x = 4$?

(12)

12. Let $f$ be a function that has derivates of all orders for all real numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$. Use a second-degree Taylor Polynomial to approximate $f(0.7)$.

13. The function $f$ has derivatives of all orders for all real numbers with $f(0) = 4$, $f'(0) = -3$, $f''(0) = 3$, and $f'''(0) = 2$. Let $g$ be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the third-degree Taylor Polynomial for $g$ about $x = 0$.

14.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ |
|---|---|---|---|---|
| $-4$ | 1 | $-2$ | $-4$ | 2 |

Selected values for $f(x)$ and its first three derivatives are shown in the table above. What is the approximation for the value of $f(-3)$ about $x = -4$ obtained using the third-degree Taylor Polynomial for $f$.

## Taylor Polynomial Approximations

**Test Prep**

15. Which of the following polynomial approximations is the best for $\cos(3x)$ near $x = 0$?

(A) $1 + \frac{3}{2}x$ (B) $1 - \frac{9}{2}x^2$ (C) $1 + x$ (D) $1 - \frac{9}{2}x + x^2$

16. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{6}(4 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 6$.

a. Write the second-degree Taylor Polynomial for $f$ about $t = 0$.

b. Use the results from part a to approximate $f(1)$.

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| $t$ (seconds) | 0 | 4 | 10 |
|---|---|---|---|
| $x'(t)$ meters per second | 5.0 | 5.8 | 4.0 |

17. The position of a particle moving along a straight line is modeled by $x(t)$. Selected values of $x'(t)$ are shown in the table above and the position of the particle at time $t = 10$ is $x(10) = 8$.

a. Approximate $x''(8)$ using the average rate of change of $x'(t)$ over the interval $4 \le t \le 10$. Show computations that lead to your answer.

b. Using correct units, explain the meaning of $x''(8)$ in the context of the problem.

c. Use a right Riemann sum with two subintervals to approximate $\int_0^{10} |x'(t)|\, dt$.

d. Let $s$ be a function such that the third derivative of $s$ with respect to $t$ is $(t - 3)^7$. Write the fourth-degree term of the fourth-degree Taylor Polynomial for $s$ about $t = 1$.

# BC Calculus – 10.8a Notes – Radius and Interval of Convergence of Power Series

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + a_3 (x-c)^3 + \cdots + a_n (x-c)^n$$

The domain of a power series is the set of all $x$-values for which the power series converges.

**Note! The center is always part of the domain.**

**Three ways a power series may converge:**

1.

   a.

2.

3.

The **Interval of Convergence** is the set of values for convergence. We use the Ratio Test to find the interval of convergence.

**Ratio Test for Interval of Convergence**

If you have a power series $\sum_{n=1}^{\infty} a_n$, find $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, then the series converges
- $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series converges

**Find the radius and interval of convergence.**

1. $\sum_{n=1}^{\infty} \frac{n}{3^n}(x-5)^n$

2. $\sum_{n=0}^{\infty} 3(x-2)^n$

3. $\sum_{n=1}^{\infty} \frac{(x+2)^{n+1}}{n^3}$

4. $\sum_{n=0}^{\infty} \frac{(2n)!\,x^{2n}}{n!}$

5. $\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

(16)

# 10.13 Radius and Interval of Convergence of Power Series

Calculus

**Practice**

**Find the interval of convergence for each power series.**

1. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$

2. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

4. $\sum_{n=0}^{\infty} (2n)!\left(\frac{x}{3}\right)^n$

**Find the radius of convergence for each series.**

5. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

6. $\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{2 \cdot 3^{n+1}}$

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

8. $\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{n!}$

**What are all values of *x* for which each series converges?**

9. $\sum_{n=1}^{\infty} \left(\frac{4}{x^2+1}\right)^n$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$

(18)

11. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$

12. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

## Radius and Interval of Convergence of Power Series

**Test Prep**

13. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

14. If the power series $\sum_{n=0}^{\infty} a_n (x-5)^n$ converges at $x = 8$ and diverges at $x = 10$, which of the following must be true?

I. The series converges at $x = 2$.

II. The series converges at $x = 3$.

III. The series diverges at $x = 0$.

(A) I only (B) II only (C) I and II only (D) II and III only

15. The coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ satisfy $a_0 = 6$ and $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$ for all $n \geq 1$. What is the radius of convergence?

(19)

16. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ is 5, what is the interval of convergence?

(A) $-5 < x < 5$  (B) $-5 < x \leq 5$  (C) $0 < x < 10$  (D) $0 < x \leq 10$

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17. Let $a_n = \frac{1}{n \ln n}$ for $n \geq 3$ and let $f$ be the function given by $f(x) = \frac{1}{x \ln x}$.

a. The function $f$ is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=3}^{\infty} a_n$.

(20)

b. Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$.

# BC Calculus – 10.9 Notes – Finding Taylor & Maclaurin Series for a Function

**Taylor Series**

If $f(x)$ has derivatives of all orders at $x = c$, then a Taylor Series may be formed that is equal to the function for many common functions.

If $c = 0$ it is a

## You need to know the following series:

The Taylor series of these functions are exact when we go to ∞. They must be memorized!

| Maclaurin Series for $e^x$. | Maclaurin Series for $\sin x$. |
|---|---|
| | |

## Memorize the following!

| Function | Series (expanded) | Series Notation | Int. of Conv. |
|---|---|---|---|
| $e^x =$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $-\infty < x < \infty$ |
| $\sin x =$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ | $-\infty < x < \infty$ |
| $\cos x =$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ | $-\infty < x < \infty$ |
| $\frac{1}{1+x} =$ | $1 - x + x^2 - x^3 + \cdots$ | $\sum_{n=0}^{\infty} (-1)^n x^n$ | $-1 < x < 1$ |

The function $f(x) = \frac{1}{1-x}$ is actually a geometric series.

Recall: $\sum_{n=k}^{\infty} ar^n =$

| **Find the Maclaurin Series for each of the following functions.** | |
|---|---|
| 3. $\sin x^2$ | 4. $x^2 e^x$ |

## Practice Problems:

1. What is the coefficient of $x^6$ in the Taylor Series about $x = 0$ for the function $f(x) = \frac{e^{2x^2}}{2}$?

23

2. If $f(x) = x \sin 3x$, what is the Taylor Series for $f$ about $x = 0$? Write the first four non-zero terms.

3. What is the Maclaurin Series for $\frac{1}{(1-x)^2}$? Write the first four non-zero terms.

4. What is the Maclaurin Series for the function $f(x) = \frac{1}{2}(e^x + e^{-x})$? Write the first four non-zero terms.

5. Find the Maclaurin Series for the function $f(x) = \cos\sqrt{x}$. Write the first four non-zero terms.

(24)

6. Find the Maclaurin Series for the function $f(x) = \sin 5x$. Write the first four non-zero terms.

7. What is the Taylor series expansion about $x = 0$ for the function $f(x) = \frac{\sin x}{x}$ ? Write the first four non-zero terms.

8. The sum of the series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots + \frac{3^n}{n!}$ is

(A) $\ln 3$ (B) $e^3$ (C) $\cos 3$ (D) $\sin 3$

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9. What is the sum of the series $1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \cdots + \frac{(\ln 3)^n}{n!}$ ?

10. What is the Taylor Series about $x = 0$ for the function $f(x) = 1 + x^2 + \cos x$ ? Write the first four non-zero terms.

25

11. What is the sum of the infinite series $1-\left(\frac{\pi}{2}\right)^2\left(\frac{1}{3!}\right)+\left(\frac{\pi}{2}\right)^4\left(\frac{1}{5!}\right)-\left(\frac{\pi}{2}\right)^6\left(\frac{1}{7!}\right)+\cdots+\frac{\left(\frac{\pi}{2}\right)^{2n}(-1)^n}{(2n+1)!}$?

12. Find the Maclaurin Series for the function $f(x)=e^{-3x}$. Write the first four non-zero terms.

13. Find the Maclaurin Series for the function $f(x)=\frac{\sin x^2}{x}+\cos x$. Write the first four non-zero terms.

14. Which of the following is the Maclaurin Series for the function $f(x)=x\cos 2x$?

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n}}{(2n)!}$ (B) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}$ (C) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n+1}}{(2n)!}$ (D) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n+1}}{(2n)!}$

26

## Finding Taylor or Maclaurin Series

**Test Prep**

15. The Maclaurin series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$ represents which function $f(x)$

(A) $\sin x$

(B) $-\sin x$

(C) $\frac{1}{2}(e^x - e^{-x})$

(D) $e^x - e^{-x}$

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16. The function $f$ satisfies the equation $f'(x) = f(x) + x + 1$ and $f(0) = 2$. The Taylor Series for $f$ about $x = 0$ converges to $f(x)$ for all $x$.

a. Write an equation for the line tangent to the curve of $y = f(x)$ at $x = 0$.

b. Find $f''(0)$ and find the second-degree Taylor Polynomial for $f$ about $x = 0$.

c. Find the fourth-degree Taylor Polynomial for $f$ about $x = 0$.

d. Find $f^{(n)}(0)$, the $n^{th}$ derivative of $f$ about $x = 0$, for $n \geq 2$. Use the Taylor Series for $f$ about $x = 0$ and the Taylor Series for $e^x$ about $x = 0$ to find $f(x) - 4e^x$.

(28)

## 10.9 AP Practice Problems (p. 799) – Taylor Series

1. The coefficient of $x^8$ in the Maclaurin expansion for $f(x) = (3x)^2 \cos x$ is

(A) $-\dfrac{1}{240}$ (B) $-\dfrac{1}{80}$ (C) $\dfrac{1}{80}$ (D) $\dfrac{3}{8}$

2. The Maclaurin expansion for $f(x) = e^{x/3}$ is

(A) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{3^k}$ (B) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{3k!}$

(C) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{3^k k!}$ (D) $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{(3k)!}$

3. The Maclaurin expansion of $f(x) = \dfrac{e^{2x} - 1}{x}$ is

(A) $\displaystyle\sum_{k=0}^{\infty} \frac{2^k x^k}{k!}$ (B) $\displaystyle\sum_{k=0}^{\infty} \frac{2^{k+1} x^k}{(k+1)!}$

(C) $\displaystyle\sum_{k=0}^{\infty} \frac{(2x)^k}{(k+1)!}$ (D) $\displaystyle\sum_{k=0}^{\infty} \frac{2x^k}{(k+1)!}$

4. The Taylor expansion for $\int_0^x \cos t\, dt$ about $\frac{\pi}{2}$ is

(A) $\sum_{k=0}^{\infty} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$

(B) $\sum_{k=0}^{\infty} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{2k+1}}{2k+1}$

(C) $\sum_{k=0}^{\infty} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{k}}{k!}$

(D) $\sum_{k=0}^{\infty} (-1)^k \frac{\left(x - \frac{\pi}{2}\right)^{2k+1}}{(2k+1)!}$

5. What is the first nonzero term of the Maclaurin expansion of $f(x) = \ln(2x^3 + 1)$?

(A) $\ln(3x^2 + 1)$ (B) $6x^2$ (C) $2x^3$ (D) $3x^2$

30

6. The Maclaurin expansion of $f(x) = \frac{1}{(1-x)^2}$ equals $\sum_{k=0}^{\infty} a_k x^k$.

(a) Find the coefficients of the first four terms of the Maclaurin expansion.

(b) Given the Maclaurin expansion of $g(x) = \frac{2}{(1-x)^3}$ equals $\sum_{k=0}^{\infty} b_k x^k$, express $b_k$ in terms of $a_k$.

# BC Calculus – 10.8b Notes – Representing Functions as a Power Series

Recall:

| Function | Series (expanded) | Series Notation | Int. of Conv. |
|---|---|---|---|
| $e^x =$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $-\infty < x < \infty$ |
| $\sin x =$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ | $-\infty < x < \infty$ |
| $\cos x =$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ | $-\infty < x < \infty$ |
| $\frac{1}{1+x} =$ | $1 - x + x^2 - x^3 + \cdots$ | $\sum_{n=0}^{\infty} (-1)^n x^n$ | $-1 < x < 1$ |

1. If $f(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$ then $f'(x) =$

2. Write the first 4 nonzero terms for the Maclaurin series that represents $\int_0^x \sin(t^7)\,dt$.

(32)

## Practice Problems:

1. What is the coefficient of $x^2$ in the Taylor Series for the function $f(x) = \sin^2 x$ about $x = 0$?

2. If the function $f$ is defined as $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$, then what is $f'(x)$? Write the first four nonzero terms and the general term.

3. Use the power series expansion for $\cos x^6$ to evaluate the integral $\int_0^x \cos t^6 \, dt$. Write the first four nonzero terms and the general term.

4. For $x > 0$, the power series defined by $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!}$ converges to which of the following?

(A) $\cos x$ (B) $\sin x$ (C) $\frac{\sin x}{x}$ (D) $e^x - e^{x^2}$

5. It is known that the Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Use this fact to assist in finding the first four nonzero terms and the general term for the power series expansion for the function $\frac{x^2}{1-x^2}$.

6. Let $f$ be the function with initial condition $f(0) = 0$ and derivative $f'(x) = \frac{1}{1+x^7}$. Write the first four nonzero terms of the Maclaurin series for the function $f$.

7. Find the Maclaurin series for the function $f(x) = e^{3x}$. Write the first four nonzero terms and the general term.

8. If a function has the derivative $f'(x) = \sin(x^2)$ and initial conditions $f(0) = 0$, write the first four nonzero terms of the Maclaurin series for $f$.

(34)

9. The function $f$ has derivatives of all orders and the Maclaurin series for the function $f$ is given by $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3}$. Find the Maclaurin series for the derivative $f'(x)$. Write the first four nonzero terms and the general term.

10. Let the function $f$ be defined by $f(x) = \frac{1}{1-x}$. Find the Maclaurin series for the derivative $f'$. Write the first four nonzero terms and the general term.

11. Find the second-degree Taylor Polynomial for the function $f(x) = \frac{\cos x}{1-x}$ about $x = 0$.

12. What is the coefficient of $x^2$ in the Maclaurin series for the function $f(x) = \left(\frac{1}{1+x}\right)^2$?

13. Find the Maclaurin series for the function $f(x) = x\cos x^2$. Write the first four nonzero terms and the general term.

14. Given that $f$ is a function that has derivatives of all orders and $f(1) = 3, f'(1) = -2, f''(1) = 2$, and $f'''(1) = 4$. Write the second-degree Taylor Polynomial for the derivative $f'$ about $x = 1$ and use it to find the approximate value of $f'(1.2)$.

15. Let the fourth-degree Taylor Polynomial be defined by $T = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ for the function $f$ about $x = 4$. Find the third-degree Taylor Polynomial for $f'$ about $x = 4$ and then use it to approximate $f'(4.2)$.

## Representing Functions as Power Series

**Test Prep**

16. Given a function defined by $f(x) = \frac{\cos(2x)-1}{x^2}$ for $x \neq 0$ and is continuous for all real numbers $x$.
    a. What is the limit of the function $f(x)$ as $x$ approaches 0?

36

b. Write the first four nonzero terms and the general term of the power series that represents the function $h(x) = \cos 2x$

c. Use the results from part (b) to write the first three nonzero terms for $f(x) = \frac{\cos(2x)-1}{x^2}$.

d. Use the results from part (c) to determine if the function $f(x) = \frac{\cos(2x)-1}{x^2}$ has a relative maximum, a relative minimum or neither at $x = 0$. Justify your answer.

## 10.8 AP Practice Problems (p. 788-789) – Power Series

1. If $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = -8$, then which of the following must be true?

   I. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = 8$

   II. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = 0$

   III. $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = -6$

   (A) I only (B) II only
   (C) II and III only (D) I, II, and III

2. If the radius of convergence of the series $\sum_{k=0}^{\infty} a_k (x-3)^k$ is 2, then which of the following must be true?

   I. The series converges for $x = 1$.
   II. The series converges for $x = 2$.
   III. The series converges for $x = 5$.

   (A) I only (B) II only
   (C) I and III only (D) I, II, and III

38

3. The interval of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x+3)^k}{k}$ is

(A) $(-4, -2)$ (B) $[-4, -2)$
(C) $[-4, -2]$ (D) $x = -3$ only

4. Which series has an interval of convergence $[-1, 1]$?

I. $\sum_{k=1}^{\infty} kx^k$ II. $\sum_{k=1}^{\infty} \frac{x^k}{k}$ III. $\sum_{k=1}^{\infty} \frac{x^k}{k^3}$

(A) I only (B) III only
(C) I and III only (D) II and III only

5. Find all numbers $x$ for which the power series $\sum_{k=0}^{\infty} \frac{x^k}{3^k}$ converges.

(A) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (B) $[-1, 1]$ (C) $(-3, 3)$ (D) $[-3, 3]$

6. The power series representation of a function $f$ is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Then the power series for $f'(x)$ is

(A) $\sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$ (B) $\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}$

(C) $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k+1)!}$ (D) $\sum_{k=1}^{\infty} \frac{kx^{k-1}}{(k-1)!}$

7. (a) Find the radius of convergence for the power series

$$\sum_{k=1}^{\infty} \frac{(x-2)^k}{3^k}$$

(b) What is the interval of convergence for the power series?

]

40

8. (a) Use the power series
$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1}$, $-1 < x \leq 1$, to find the power series representation for $f(x) = \ln(1+x^2)$.

(b) Use properties of power series to find the derivative of $f$.

(c) What is the interval of convergence for $f'$?

# BC Calculus – 10.10b Notes – Lagrange Error Bound

Exact value = Approximate value + Remainder

Error:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!} + R(x)$$

**Lagrange Error Bound**

Let $f(x)$ be differentiable through the order $n+1$. The error between the Taylor Polynomial and $f(x)$ is bounded by:

$$|R_n(x)| \leq$$

where $z$ is some number between $c$ and $x$.

1. The fourth degree Maclaurin polynomial for $\cos x$ is given by $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. If this polynomial is used to approximate $\cos(0.2)$, what is the Lagrange error bound?

42

**<u>LaGrange Error Bound Summary</u>:**

$$|R_n(x)| = \left|\frac{Max\left[f^{(n+1)}(z)\right]}{(n+1)!}(x-c)^{(n+1)}\right|$$

- $\boldsymbol{R_n(x)}$ is the error bound or the approximation between the Taylor polynomial & the function f(x)
- $\boldsymbol{x} = the\ given\ value$
- $\boldsymbol{c} = the\ center\ of\ the\ function. For\ instance, a\ MacLaurin\ Series\ would\ be\ c = 0$
- $\boldsymbol{n} = the\ degree\ of\ the\ polynomial$
- $\boldsymbol{z} = some\ unknown\ x - value\ between\ c\ and\ x\ where\ the\ maximum\ y - value\ \ of\ (n+1)^{st}\ function\ lies.$

When applying Taylor's Formula, we would not expected to be able to find the exact value of $z$. (If we could do this, then an approximation would not be necessary). Rather, we are merely interested in a safe upper bound (maximum value) for the $|f^{(n+1)}(z)|$ between x and c, from which we will be able to tell how large the remainder $R_n(x)$ is.

- We want to maximize the (n+1)[st] derivative on the interval from [x , c] or [c,x]. The maximum error bound is the worst case scenario for the interval in which our actual approximation can live

2. Use a third degree Taylor polynomial on the interval $[0, 1]$ for $e^x$ centered about $x = 0$ to approximate $e^1$. What is the error bound of this approximation?

3. What is the smallest order Taylor Polynomial centered at $x = 1$ which will approximate $e^{x-1}$ on the interval $[0, 3]$ with a Lagrange error bound less than 1?

## Practice Problems:

1. The third Maclaurin polynomial for $\sin x$ is given by $p(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.1)$, what is the Lagrange error bound?

44

2. If the Taylor Polynomial for approximating $\cos x$ is given by $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, what is the upper bound for the error in the approximation of $\cos(0.3)$?

3. If the Taylor Polynomial about $x = 0$ for the approximation of $e^x$ is given by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$, what is the upper bound for the error in the approximation of $e$?

4. Let $f$ be a function that has derivatives of all orders for all real numbers and let $P_3(x)$ be the third-degree Taylor Polynomial for $f$ about $x = 0$. $\left|f^{(n)}(x)\right| \leq \frac{n}{n+1}$, for $1 \leq n \leq 5$ and all values of $x$. Of the following, which is the smallest value of $k$ for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \leq k$?

(A) $\frac{5}{6}$ (B) $\frac{5}{6} * \frac{1}{5!}$ (C) $\frac{5}{6} * \frac{1}{4!}$ (D) $\frac{4}{5} * \frac{1}{4!}$

5. The function $f$ has derivatives of all orders for all real numbers, $f^{(4)}(x) = e^{\cos x}$. If the third-degree Taylor Polynomial for $f$ about $x = 0$ is used to approximate $f$ on the interval [0,1], what is the Lagrange error bound?

45

6. The Taylor series for a function $f$ about $x = 3$ is given by $\sum_{n=0}^{\infty}(-1)^n \frac{3n+1}{2^n}(x-3)^n$ and converges to $f$ for $0 \leq x \leq 5$. If the third-degree Taylor Polynomial for $f$ about $x = 3$ is used to approximate $f\left(\frac{13}{4}\right)$, what is the alternating series error bound?

7. Let $f$ be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. $T_4(x)$ is the fourth-degree Taylor Polynomial for $f$ about $x = c$ such that $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$. Based on the Lagrange error bound, $f(x) - T_4(x)$ must equal which of the following?

(A) $x$ (B) $(x-c)^5$ (C) $a_5(x-c)^5$ (D) $\frac{a_5(x-c)^5}{5!}$

---

8. Let $P(x)$ be the sixth-degree Taylor Polynomial for a function $f$ about $x = 0$. Information about the maximum of the absolute value of selected derivatives of $f$ over the interval $0 \leq x \leq 1.5$ is given below.

$\max_{0\leq x\leq 1.5}\left|f^{(5)}(x)\right| = 9.3$ $\max_{0\leq x\leq 1.5}\left|f^{(6)}(x)\right| = 62.1$ $\max_{0\leq x\leq 1.5}\left|f^{(7)}(x)\right| = 481.3$

What is the smallest value of $k$ for which the Lagrange error bound guarantees that $|f(1.5) - P(1.5)| \leq k$ ?

9. The function $f$ has derivatives of all orders for all real numbers. Values of $f$ and its first four derivatives at $x = 2$ are given in the table.

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ | $f^{(4)}(x)$ |
|---|---|---|---|---|---|
| 2 | 6 | −12 | 18 | −24 | 34 |

a. Write the third-degree Taylor Polynomial for $f$ about $x = 2$, and use it to approximate $f(1.5)$.

46

b. The fourth derivative of $f$ satisfies the inequality $|f^{(4)}(x)| \le 48$ for all $x > 1$. Use the Lagrange error bound to show that the approximation found in part (a) differs from $f(1.5)$ by no more than $\frac{1}{8}$.

10. Let $h$ be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of $h$ are given for $x = 3$. Use the Lagrange error bound to show that the third-degree Taylor polynomial for $h$ about $x = 3$ approximates $h(2.9)$ with an error less than $3 \times 10^{-4}$.

| $x$ | $h(x)$ | $h'(x)$ | $h''(x)$ | $h'''(x)$ | $h^{(4)}(x)$ |
|---|---|---|---|---|---|
| 3 | 317 | $\frac{753}{4}$ | $\frac{1383}{4}$ | $\frac{3483}{8}$ | $\frac{1125}{16}$ |

## Lagrange Error Bound

**Test Prep**

11. **Calculator allowed.**

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ | $f^{(4)}(x)$ |
|---|---|---|---|---|---|
| 3 | 4 | $-8$ | 14 | $-22$ | 30 |

The function $f$ has derivatives of all orders for all real numbers. Values of $f$ and its first four derivatives at $x = 3$ are given in the table.

a. Write an equation for the line tangent to the graph of $f$ at $x = 3$ and use it to approximate $f(2.5)$.

47

b. Write the third-degree Taylor polynomial for $f$ about $x = 3$, and use it to approximate $f(2.5)$.

c. Is there enough information to determine whether $f$ has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

d. The fourth derivative of $f$ satisfies the inequality $\left|f^{(4)}(x)\right| \leq 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

e. What is the coefficient of the $(x - 3)^3$ term in the Taylor series for $f'$, the derivative of $f$, about $x = 3$?

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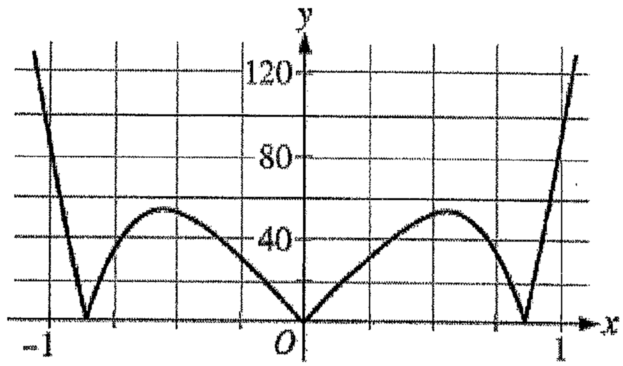
### Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.




Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for $f$ about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for $f$ about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

49

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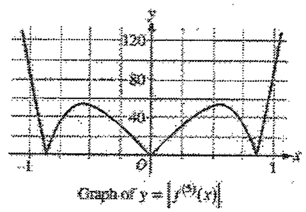
### Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for $f$ about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for $f$ about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$. $* \; R_n(x) = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = 1 + x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} - \frac{x^6}{6!}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6} - \frac{x^6}{720}$

$1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{120x^6}{720} - \frac{1x^6}{720}$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121}{6!}x^6$

c) Since $\frac{f^{(6)}(0)}{6!}x^6 = \frac{-121}{6!}x^6$,

$\boxed{f^{(6)}(0) = -121}$

d) $R_4(x) = \left|P_4(x) - f(x)\right| = \left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| = \left|\frac{f^{(5)}(z)}{5!}(x-c)^5\right| \le \left|\frac{40}{5!}\left(\frac{1}{4} - 0\right)^5\right|$

$x = 1/4$, $c = 0$; max y-value in interval $0 < x < 1/4$

$= \left|\frac{40}{5!}\left(\frac{1}{4}\right)^5\right| = \frac{40}{120 \cdot 4^5} = \boxed{\frac{1}{3072} < \frac{1}{3000}}$

$= \frac{1}{3 \cdot 4^5} = \frac{1}{3 \cdot 1024}$

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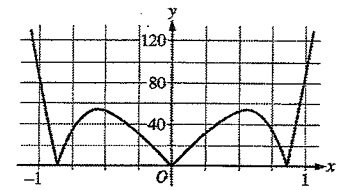
### Question 6

Let $f(x) = \sin(x^2) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for $f$ about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for $f$ about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

| | |
|---|---|
| (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$<br>$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$ | 3 : { 1 : series for $\sin x$<br>2 : series for $\sin(x^2)$ |
| (b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$<br>$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$ | 3 : { 1 : series for $\cos x$<br>2 : series for $f(x)$ |
| (c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of $x^6$ in the Taylor series for $f$ about $x = 0$. Therefore $f^{(6)}(0) = -121$. | 1 : answer |
| (d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.<br>Therefore<br>$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$. | 2 : { 1 : form of the error bound<br>1 : analysis |

50

## 10.10 AP Practice Problems (p.806) – Taylor Polynomial Approximations & Lagrange Error Bound

1. Use the Taylor polynomial $P_4(x)$ of $y = \cos x$ at 0 to approximate $\cos 0.2$.

   (A) 1 (B) 0.980 (C) 0.803 (D) 0.801

2. If a function $f$ can be represented by the Taylor polynomial

$$P_3(x) = 4 + 2(x-2) + 3(x-2)^2 + \frac{1}{2}(x-2)^3$$

   then $f''(2) =$

   (A) 1 (B) 2 (C) 3 (D) 6

3. If $f(-1) = 4$, $f'(-1) = -3$, $f''(-1) = 3$, and $f'''(-1) = 2$ then the Taylor polynomial $P_3(x)$ of degree 3 of $f$ at $-1$ is

   (A) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{2}{3}(x+1)^3$

   (B) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 - \frac{1}{3}(x+1)^3$

   (C) $P_3(x) = 4 - 3(x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

   (D) $P_3(x) = 4 - 3(x+1) + \frac{3}{2}(x+1)^2 + \frac{1}{3}(x+1)^3$

4. The polynomial $f(x) = x^3 - x^2 - 4$ expressed in powers of $(x + 1)$ equals

(A) $-4 + (x + 1) + 4(x + 1)^2 + 6(x + 1)^3$

(B) $-6 + 5(x + 1) + 4(x + 1)^2 + (x + 1)^3$

(C) $-6 + 5(x + 1) - 8(x + 1)^2 + 6(x + 1)^3$

(D) $-6 + 5(x + 1) - 4(x + 1)^2 + (x + 1)^3$

5. The Lagrange error bound in using a Taylor polynomial of degree 5 for $f(x) = \sin x$ at $\frac{2\pi}{3}$ to approximate $\sin 2$ is no more than

(A) $\dfrac{\left|\frac{2\pi}{3} - 2\right|^5}{5!}$

(B) $\dfrac{\left|\frac{2\pi}{3} - 2\right|^6}{6!}$

(C) $\dfrac{\left|\cos\frac{2\pi}{3}\right|\left|\frac{2\pi}{3} - 2\right|^6}{6!}$

(D) $\dfrac{\left|\sin\frac{2\pi}{3}\right|\left|\frac{2\pi}{3} - 2\right|^6}{6!}$

52

6. (a) Write the Maclaurin series for $f(x) = e^{-x}$.

   (b) Approximate $e^{-0.1}$ with the Taylor polynomial $P_2(x)$ of $f(x) = e^{-x}$.

   (c) How many terms are necessary to approximate $e^{-0.1}$ with an error less than or equal to 0.00001?

7. The standard normal probability distribution density function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is defined for all real numbers.

(a) Write the first four nonzero terms of the Maclaurin expansion for $f$.

(b) Use the Maclaurin expansion from (a) and properties of a power series to approximate $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$.

(c) Find the maximum error in using the first four terms of the Maclaurin expansion to approximate $\int_0^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$.

(d) Justify the method used to find the error in (c).

54

# Ch. 10 Unit Review AP Practice Problems (p.813) – Infinite Series

1. Suppose $f(2) = 3$; $f'(2) = 0$; $f''(2) = 5$; $f'''(2) = -4$; and $f^{(4)}(2) = -2$. Then the Taylor polynomial $P_4(x)$ of degree 4 of $f$ at 2 is

(A) $P_4(x) = 3(x-2) + \frac{5}{2}(x-2)^2 - \frac{4}{6}(x-2)^3 - \frac{2}{24}(x-2)^4$

(B) $P_4(x) = 3 + \frac{5}{2}(x-2)^2 - \frac{2}{3}(x-2)^3 - \frac{1}{12}(x-2)^4$

(C) $P_4(x) = 3 + (x-2) + \frac{5}{2}(x-2)^2 - \frac{2}{3}(x-2)^3 + \frac{1}{12}(x-2)^4$

(D) $P_4(x) = 3 + \frac{5}{2!}(x-2)^2 - \frac{4}{3!}(x-2)^3 + \frac{2}{4!}(x-2)^4$

2. If $0 < a_k \le b_k$ for all $k$ and $\sum_{k=1}^{\infty} a_k$ diverges, then which statement must be false?

(A) $\lim_{n\to\infty} a_n = 0$ (B) $\lim_{n\to\infty} b_n = 0$

(C) $\sum_{k=1}^{\infty} b_k = 1$ (D) $\sum_{k=1}^{\infty} (-1)^k a_k$ diverges

3. Which of the following series converge?

I. $\sum_{k=1}^{\infty} k^{-3/2}$ II. $\sum_{k=1}^{\infty} k^{-1}$ III. $\sum_{k=1}^{\infty} 2^{1-k}$

(A) I only (B) III only

(C) I and III only (D) I, II, and III

4. Determine whether the infinite series $\sum_{k=1}^{\infty}\left(-\frac{5}{6}\right)^{k-1}$ converges or diverges. If the series converges, find its sum $S$.

(A) converges; $S = -\frac{5}{11}$ (B) converges; $S = \frac{6}{11}$

(C) converges; $S = 6$ (D) The series diverges.

5. The interval of convergence of $\sum_{k=1}^{\infty}\frac{(x-2)^k}{k^2}$ is

(A) $-1 < x < 1$ (B) $-1 < x \leq 1$

(C) $1 \leq x < 3$ (D) $1 \leq x \leq 3$

6. For which integer $n$ do all three infinite series converge?

$$\sum_{k=1}^{\infty} k^{-n/3} \qquad \sum_{k=1}^{\infty}\frac{(-1)^{nk}}{k} \qquad \sum_{k=1}^{\infty}\left(\frac{n}{6}\right)^k$$

(A) 2 (B) 3 (C) 4 (D) 5

7. If a function $f$ is continuous, positive, and decreasing on the interval $[1, \infty)$ and if $a_k = f(k)$ for all positive integers $k$, then the infinite series

(A) $\sum_{k=1}^{\infty} a_k$ converges.

(B) $\sum_{k=1}^{\infty} a_k = \int_1^{\infty} f(x)dx$ if $\int_1^{\infty} f(x)dx$ converges.

(C) $\sum_{k=1}^{\infty} a_k = f(x)$

(D) $\sum_{k=1}^{\infty} a_k$ converges if $\int_1^{\infty} f(x)dx$ converges.

8. $\sum_{k=1}^{\infty} \frac{3^{k+1} + 4k!}{3^k \cdot k!} =$

(A) $e + 2$ (B) $3e - 1$ (C) $3e + \frac{8}{3}$ (D) $3e + 2$

9. Find bounds for the sum of $\sum_{k=1}^{\infty} \frac{3}{(2k)^4}$.

(A) $\frac{1}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{1}{4}$ (B) $\frac{1}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{17}{16}$

(C) $\frac{3}{16} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{4}{3}$ (D) $\frac{1}{3} < \sum_{k=1}^{\infty} \frac{3}{(2k)^4} < \frac{4}{3}$

10. Which of the series diverge?

I. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots$

II. $2 + \frac{4}{2^{3/2}} + \frac{6}{3^{3/2}} + \frac{8}{4^{3/2}} + \cdots$

III. $6 - 4 - \frac{8}{3} - \frac{16}{9} - \frac{32}{27} - \cdots$

(A) I only (B) II only

(C) I and II only (D) I and III only

11. The interval of convergence of the power series $\sum_{k=0}^{\infty} \left(\frac{x-1}{4}\right)^k$ is

(A) $-1 \le x \le 1$ (B) $-4 < x < 4$

(C) $-4 < x \le 4$ (D) $-3 < x < 5$

12. Which series is the Maclaurin expansion of $f(x) = x^2 \sin x$?

(A) $x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \frac{x^8}{6!} + \cdots$

(B) $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

(C) $x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \cdots$

(D) $\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$

58

13. Determine whether the series $\sum_{k=1}^{\infty} \frac{4^{k+2}}{5^k}$ converges. If it converges, its sum equals

(A) converges; 5
(B) converges; 64
(C) converges; 80
(D) The series diverges.

14. Determine whether the series $\sum_{k=1}^{\infty} \frac{6}{(k+1)(k+2)}$ converges. If it converges, its sum equals

(A) converges; 1
(B) converges; 3
(C) converges; 6
(D) The series diverges.

15. A Maclaurin polynomial for $f(x) = e^x$ is used to approximate $e^{1/3}$. What is the degree of the Maclaurin polynomial needed to ensure that the Lagrange error bound is less than 0.00001?

(A) 3 (B) 5 (C) 7 (D) 9

16. A fourth degree Taylor polynomial for $f(x) = \sin x$ at $\frac{\pi}{6}$ is used to approximate $\sin 35°$. The Lagrange error bound guarantees the error in using the approximation is less than

(A) $\frac{1}{4!}\left(\frac{\pi}{36}\right)^4$ (B) $\frac{1}{5!}\left(\frac{\pi}{36}\right)^5$

(C) $\frac{1}{5!}\left(\frac{\pi}{6}\right)^5$ (D) $\frac{1}{7!}\left(\frac{\pi}{6}\right)^7$

17. Determine whether the infinite series $\sum_{k=1}^{\infty} \frac{(\ln k)^2}{k}$ converges or diverges. Be sure to show all your work.

18. Determine whether the infinite series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sqrt{k}}{k+1}$ is absolutely convergent, conditionally convergent, or divergent. Show your work.

19. (a) Write the first five nonzero terms of the Maclaurin expansion of $f(x) = \tan^{-1} x$.

(b) Use properties of power series to obtain the Maclaurin series for $g(x) = \int_0^x \tan^{-1} t \, dt$.

(c) Using the fact that the radius of convergence $R$ of the Maclaurin series representation for $f(x) = \tan^{-1} x$ is 1, find the interval of convergence of the series representation of $g$.

(d) Use the result from (b) to approximate $\int_0^{1/4} \tan^{-1} x \, dx$ so the error is less than 0.001.

**Taylor Polynomial** is a polynomial that will approximate other function's values in a region that is nearby the "center"
*a tangent line is essentially a first degree taylor polynomial.

**$n^{th}$ degree taylor polynomial:**

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

**Alternating Series Remainder:**
Suppose an alternating series converges by AST. If the Series has Sum S, then $|R_n| = |S - S_n| \le |a_{n+1}|$

*This means that the **maximum error** for the $n^{th}$ term partial Sum $S_n$ is no greater than the absolute value of the first unused term $a_{n+1}$

**Taylor Series:** A General method for writing a power series representation for a function.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f^{(n)}$ represents the $n^{th}$ derivative evaluated at f.

**Power Series:** Written in form $\sum_{n=0}^{\infty} a_n (x-c)^n$ where c and $a_n$ (coefficients) are numbers:

61

*Taylor and Maclaurin series are special cases of power series
For a power series centered at c, precisely one of the following is true:
1) The series converges only at c (ALL power series converge at least at their center) (Radius of convergence = 0)
2) The series converges for all x (function and infinite series have exact same values everywhere) → Radius = ∞
3) The series converges within a certain Radius of Convergence such that series converges for $|x - c| < R$
→The **interval of Convergence** (I.O.C.) is **[ ( c − R , c + R ) ]**
*Be sure to TEST convergence of endpoints
*Typically, you want to use the **RATIO TEST** to determine Radius of Convergence

**Maclaurin Series**: is the special case of Taylor series when **c = 0**.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

**Special Maclaurin Series:**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$ IOC: All Reals

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!}$$ IOC: All Reals

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^n}{n!}$$ IOC: All Reals

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)}$$ IOC: $-1 \le x \le 1$

**Geometric Series** below based on

$$S = \frac{a_1}{1-r}$$ IOC: $-1 < x < 1$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \ldots (-1)^n x^n + \ldots$$

IOC: $-1 < x < 1$

$$\frac{1}{x} = \frac{1}{1-[-(x-1)]} = 1 - (x-1) + (x-1)^2 - \ldots (-1)^n (x-1)^n$$

IOC: $0 < x < 2$

$$\ln x = \int \frac{1}{x}\,dx = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \ldots \frac{(-1)^{n-1}(x-1)^n}{n}$$

IOC: $0 < x \le 2$

**LaGrange Error Bound** *This is similar to the Alternating Series Remainder. However, this method offers a way to determine the maximum error (remainder) when we do a Taylor polynomial approximation using a certain number of terms for a specific function.

$$R_n(x) = \left| \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right| \le \left| \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}(x-c)^{n+1} \right|$$

* The remainder for an $n^{th}$ degree polynomial is found by taking the $(n+1)^{st}$ (**first unused**) derivative at "z" *We are not expected to find the exact value of z. (If we could, then an approximation would not be necessary) *We want to maximize the $(n+1)^{st}$ derivative on the interval from [x , c] in order to find a safe upper bound for the $\left|f^{(n+1)}(z)\right|$ *The maximum error bound is the worst case scenario for the interval in which our actual approximation can live. **College Board will provide strictly increasing and decreasing functions. (So we only have to choose between f( c) and f(x) (the endpoints). This will allow us to determine the max value much more accurately.

**Alternating Series Remainder**:
Suppose an alternating series converges by AST (such that $\lim_{n\to\infty} a_n = 0$ and $a_n$ is decreasing), then

$$|R_n| = |S - S_n| \le |a_{n+1}|$$

*This means that the **maximum error** for the $n^{th}$ term partial Sum $S_n$ is no greater than the absolute value of the first unused term $a_{n+1}$

2