Key

*Must be a converging alternating series!

BC Calculus – 10.5b Notes – Alternating Series Error Bound

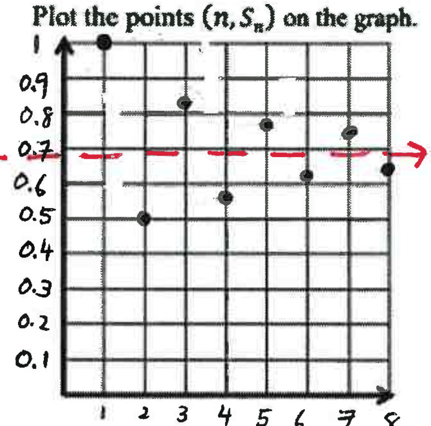
Use the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ to fill in the table below.

		n	1	2	3	4	5	6	7	8
terms	a_n	fractions	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$
		decimals	1	−0.5	$0.\overline{33}$	−0.25	0.2	$-0.1\overline{66}$	0.143	−0.125
partial Sum	S_n	fractions	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{47}{60}$	$\frac{37}{60}$	$\frac{319}{420}$	$\frac{533}{840}$
		decimals	1	0.5	0.833	0.583	0.783	0.616	0.759	0.634

partial Sums

*This series will keep alternating and bouncing above and below the line until eventually the partial sums approach $\ln 2$ (closer to the line)

Plot the points (n, S_n) on the graph.



*The actual Sum of this Series is $\ln 2 \approx 0.6931$

	R1	R2						
Error: $\lvert S - S_n\rvert$	$\ln 2 - 1$ 0.3068	$\lvert\ln 2 - 1/2\rvert$ 0.193	$\lvert\ln 2 - 5/6\rvert$ 0.14	$\lvert\ln 2 - 7/12\rvert$ 0.1098	$\lvert\ln 2 - 47/60\rvert$ 0.0901	0.0764	0.0663	0.0586
$\leq$	$\leq a_2$ (0.5)	$\leq a_3$ (0.33)	≤ 0.25	≤ 0.2	≤ 0.166	≤ 0.143	≤ 0.125	

Alternating Series Error Bound

If you have an alternating series that converges, we can approximate the sum of the series!

$$|S - S_n| = |R_n| \leq |a_{n+1}|$$

Actual Remainder many times is not known to us.

Error bound is the next term (can be found)

S: Sum of the series
S_n: Partial sum
R_n: Remainder (or error)
$R_n = S - S_n$
a_{n+1} = next term (Error Bound)

*Error Bound is the boundary of how far off your actual error is.

1. Determine the number of terms required to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ with an error less than 10^{-3}.

$$|S-S_n| \leq |a_{n+1}|$$

*what makes $\frac{1}{n+1} < 10^{-3}$

or $\frac{1}{n+1} < 0.001$

$$\frac{1}{n+1} < \frac{1}{1000}$$

$$1000 < n+1$$

$$999 < n$$

If $n > 999$ terms, the partial sum's approximation is off by no more than $\frac{1}{1000}$ or 0.001

2. If the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n+2}$ is approximated by the partial sum with 10 terms, what is the alternating series $\underbrace{\text{error bound}}_{a_{n+1}}$?

$$a_{11} = \left|\frac{1}{5(11)+2}\right| = \frac{1}{57}$$

The partial sum S_{10} is off by no more than $\frac{1}{57}$ of the Actual sum

3. **Calculator active.** Approximate an interval of the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n^2}$ using the Alternating Series Error Bound for the first 5 terms.

*Math→0 (in calculator)

Approx: $S_5 = \sum_{n=1}^{5} a_n = 3.35\overline{4}$

$$3.354 - 0.111 \leq S \leq 3.354 + 0.111$$

$$\boxed{3.24\overline{3} \leq S \leq 3.46\overline{5}}$$

$|a_6| = \left|\frac{4}{6^2}\right| = \frac{1}{9} = 0.1\overline{1}$ is the boundary for how far off our approximation is.

4. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$. Show that $\underset{\uparrow\ S_n}{1 - \frac{1}{3!}}$ approximates $\underset{\uparrow\ S}{f(1)}$ with an error less than 0.01.

$$|S-S_n| \leq |a_{n+1}|$$

$$\left|S-\left(1-\frac{1}{3!}\right)\right| \leq \frac{1^4}{5!} \leq \frac{1}{120} \approx 0.0083\overline{3} \leq 0.01 ✓$$

$$\frac{1}{120} \leq \frac{1}{100} ✓$$

[redacted] Alternating Series Error Bound

Calculus

Practice

A calculator may be used on all problems in this practice. For 1-2, approximate an interval of the sum of the alternating series using the Alternating Series Error Bound for the first 6 terms.

*select Math→0

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

$\sum_{x=1}^{6} a_n \approx 0.185185$

$|a_7| = \frac{7}{3^7} \approx 0.0032$

Sum is 0.185185 ± 0.0032

$\boxed{0.1819 \leq S \leq 0.188}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n+2)}$

$\sum_{x=1}^{6} a_n \approx 1.14046$

$|a_7| = \frac{4}{\ln(9)} \approx 1.8204$

$S = 1.14046 \pm 1.8204$

$\boxed{-0.68 \leq S \leq 2.9609}$

3. Determine the number of terms needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with an error less than 10^{-3}.

$|a_{n+1}| < 10^{-3}$

$\frac{1}{(n+1)^2} < \frac{1}{1000}$

$(n+1)^2 > 1000$

$n+1 > \sqrt{1000}$

$n+1 > 31.6227$

$n > 30.6227$

$\boxed{n = 31 \text{ or more terms}}$

4. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges to S. Using the alternating series bound, what is the least number of terms that must be summed to guarantee a partial sum that is within 0.05 of S?

$|a_{n+1}| < 0.05$

$\frac{1}{\sqrt{n+1}} < 0.05$

$\sqrt{n+1} > \frac{1}{0.05}$

$\sqrt{n+1} > 20$

$n+1 > 400$

$n > 399$

(A) 20 (B) 55 (C) 399 $\boxed{\text{(D) 400}}$

5. If the infinite series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n}$ is approximated by $P_k = \sum_{n=1}^{k}(-1)^{n+1}\frac{4}{n}$, what is the least value of k for which the alternating series error bound guarantees that $|S - P_k| < \frac{7}{100}$?

error less than $\frac{7}{100}$

$\frac{4}{k+1} < \frac{7}{100}$

$7(k+1) > 400$

$k+1 > \frac{400}{7}$

$k > \frac{400}{7} - 1$

$k > 56.1428$

(A) 55 (B) 56 $\boxed{\text{(C) } 57}$ (D) 60

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6. If the series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ is approximated by the partial sum $S_k = \sum_{n=1}^{k}(-1)^{n+1}\frac{1}{n^3}$, what is the least value of k for which the alternating series error bound guarantees that $|S - S_k| \le \frac{7}{10000}$?

error less than $\frac{7}{10000}$

$\frac{1}{(k+1)^3} < \frac{7}{10000}$

$7(k+1)^3 > 10000$

$(k+1)^3 > \frac{10000}{7}$

$k+1 > \sqrt[3]{\frac{10000}{7}}$

$k > 10.262$

(A) 10 $\boxed{\text{(B) } 11}$ (C) 12 (D) 13

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7. The series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges by the alternating series test. If $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ is the nth partial sum of the series, which of the following statements must be true?

(A) $\lim_{n\to\infty} S_n = 0$ (B) $\lim_{n\to\infty} a_n = S$ (C) $|S - S_{20}| \le a_{26}$ $\boxed{\text{(D) } |S - S_{25}| \le a_{26}}$

8. If the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{5n+1}$ is approximated by the partials sum with 15 terms, what is the alternating series error bound?

$$|a_{n+1}| = \frac{1}{5(n+1)+1} \rightarrow \frac{1}{5(16)+1} = \boxed{\frac{1}{81}}$$

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{76}$ $\boxed{\text{(D) } \frac{1}{81}}$

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9. The function f is defined by the power series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x}{(2n+1)!}$ for all real numbers x. Show that $1 - \frac{1}{3!} + \frac{1}{5!}$ approximates $f(1)$ with an error less than $\frac{1}{4000}$.

$$|S - S_n| < |a_{n+1}|$$

$$\left| f(1) - \left[1 - \tfrac{1}{3!} + \tfrac{1}{5!}\right] \right| < \frac{1}{7!}$$

$\frac{1}{7!} = \frac{1}{5040}$ which is less than $\frac{1}{4000}$

next term in the series
(error bound for partial sum $(1 - \frac{1}{3!} + \frac{1}{5!})$)

[redacted] Alternating Series Error Bound

Test Prep

10. **Calculator active!** Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

a. Use the first three terms of the series to approximate $f\left(-\frac{1}{3}\right)$.

$$f(-1/3) \approx \sum_{n=1}^{3} \frac{(-1/3)^n n^n}{n!} = -0.2778$$

b. How far off is this estimate from the value of $f\left(-\frac{1}{3}\right)$? Justify your answer.

$$a_4 = \frac{(-1/3)^4 \cdot 4^4}{4!} = 0.13168$$

The estimate of -0.2778 is off by at most 0.13168 or less.
a_4 is the error bound for the partial sum S_3
(the next unused term)

11. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is approximated with the series $\sum_{n=1}^{7}(-1)^{n+1}\frac{1}{n^2}$, what is the error bound?

Error Bound for partial sum S_7 is $|a_8| \leftarrow 8^{th}$ term

$$|a_8| = \frac{1}{8^2} = \boxed{\frac{1}{64}}$$