BC Calculus – 10.7 Notes – Summary of Convergence Tests

TABLE 5 Tests for Convergence and Divergence of Series

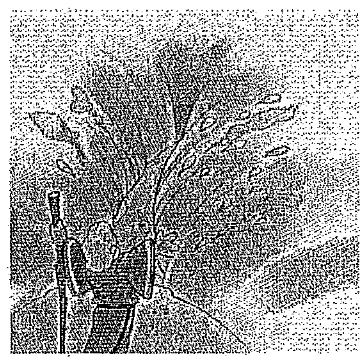
Test Name	Description	Comment
nth Term Test for Divergence for all series (p. 735)	$\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} a_n \neq 0$.	No information is obtained about convergence if $\lim_{n\to\infty} a_n = 0$.
Integral Test for series of positive terms (p. 738)	$\sum_{k=1}^{\infty} a_k$ converges (diverges) if $\int_1^{\infty} f(x)\ dx$ converges (diverges), where f is continuous, positive, and nonincreasing for $x \geq 1$; and $f(k) = a_k$ for all k.	Good to use if f is easy to integrate.
Comparison Test for Convergence for series of positive terms (p. 747)	$\sum_{k=1}^{\infty} a_k$ converges if $0 < a_k \leq b_k$ and the series $\sum_{k=1}^{\infty} b_k$ converges.	$\sum_{k=1}^{\infty} b_k$ must have positive terms and be convergent.
Comparison Test for Divergence for series of positive terms (p. 747)	$\sum_{k=1}^{\infty} a_k$ diverges if $a_k \geq c_k > 0$ and the series $\sum_{k=1}^{\infty} c_k$ diverges.	$\sum_{k=1}^{\infty} c_k$ must have positive terms and be divergent.
Limit Comparison Test for series of positive terms (p. 748)	$\sum_{k=1}^{\infty} a_k$ converges (diverges) if $\sum_{k=1}^{\infty} b_k$ converges (diverges), and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, a positive real number.	$\sum_{k=1}^{\infty} b_k$ must have positive terms, whose convergence (divergence) can be determined.
Alternating Series Test (p. 755)	$\sum_{k=1}^{\infty} (-1)^{k+1} a_k,\ a_k > 0,$ converges if • $\lim_{n\to\infty} a_n = 0$ and • the a_k are nonincreasing.	The error made by using the nth partial sum to approximate the sum S of the series is less than or equal to $\lvert a_{n+1} \rvert$.
Absolute Convergence Test (p. 759)	If $\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.	The converse is not true. That is, if $\sum_{k=1}^{\infty} \lvert a_k \rvert$ diverges, $\sum_{k=1}^{\infty} a_k$ may or may not converge.
Ratio Test for series with nonzero terms (p. 765)	$\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$. $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ or if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \infty$.	Good to use if a_n includes factorials or powers. It provides no information if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$ or if $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert \neq \infty$ does not exist.
Root Test for series with nonzero terms (p. 768)	$\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$. $\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$ or if $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = \infty$.	Good to use if a_n involves nth powers. It provides no inform. $\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = 1$.

TABLE 6 Important Series

Series Name	Series Description	Comments
Geometric series (pp. 725–726)	$\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + \cdots, \quad a \neq 0$	Converges to $\frac{a}{1-r}$ if $\lvert r\rvert < 1$; diverges if $\lvert r\rvert \geq 1$.
Harmonic series (p. 729)	$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$	Diverges.
p-series (p. 740)	$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$	Converges if $p > 1$; diverges if $0 < p \leq 1$.
k-to-the-k series (p. 747)	$\sum_{k=1}^{\infty} \frac{1}{k^k} = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$	Converges.
Alternating harmonic series (p. 756)	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$	Converges.

So, how can you remember all these tests (besides using your Jedi powers)? Try this Moses phrase:

PARTING C



P p-series: Is the series in the form $\frac{1}{n^p}$?

A Alternating series: Does the series alternate? If it does, are the terms getting smaller, and is the nth term 0?

R Ratio Test: Does the series contain things that grow very large as n increases (exponentials or factorials)?

T Telescoping series: Will all but a couple of the terms in the series cancel out?

I Integral Test: Can you easily integrate the expression that defines the series (are Dogs Cussing in Prison?)

N nth Term divergence Test: Is the nth term something other than zero?

G Geometric series: Is the series of the form $\sum_{n=0}^{\infty} ar^n$?

C Comparison Tests: Is the series ***almost*** another kind of series (e.g. p-series or geometric)? Which would be better to use: the Direct or Limit Comparison Test?

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Summary of Tests for Infinite Series Convergence

Given a series

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum_{n=0}^{\infty} a_n$$

The following is a summary of the tests that we have learned to tell if the series converges or diverges. They are listed in the order that you should apply them, unless you spot it immediately, i.e. use the first one in the list that applies to the series you are trying to test, and if that doesn't work, try again. Off you go, young Jedis. Use the Force. Remember, it is always with you, and it is mass times acceleration!

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nth-term test: (Test for Divergence only)

If $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent. If $\lim_{n\to\infty} a_n = 0$, then the series may converge or diverge, so you need to use a different test.

Geometric Series Test:

If the series has the form $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^{n}$, then the series converges if $|r| < 1$ and diverges otherwise. If the series converges, then it converges to $\frac{a_1}{1-r}$.

Integral Test:

In ***P***rison, ***D***ogs ***C***urse: If $a_n = f(n)$ is ***P***ositive, ***D***ecreasing, ***C***ontinuous function, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(n)dn$ either both converge or both diverge.

- This test is best used when you can easily integrate a_n.
- ***Careful***: If the Integral converges to a number, this is NOT the sum of the series. The series will be smaller than this number. We only know this it also converges, to what is anyone's guess.
- The maximum error, R_n, for the sum using S_n will be $0 \le R_n \le \int_n^{\infty} f(x)dx$

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p-series test:

If the series has the form $\sum \frac{1}{n^p}$, then the series converges if $p > 1$ and diverges otherwise. When $p = 1$, the series is the divergent Harmonic series.

Alternating Series Test:

If the series has the form $\sum(-1)^n a_n$, then the series converges if $0 < a_{n+1} \le a_n$ (decreasing terms) for all n, for some n, and $\lim_{n\to\infty} b_n = 0$. If either of these conditions fails, the test fails, and you need use a different test.

- if the series converges, the sum, S, lies between $S_n - a_{n+1}$ and $S_n + a_{n+1}$
- if $\left|\sum a_n\right|$ converges then $\sum a_n$ is *Absolutely Convergent*
- if $\left|\sum a_n\right|$ diverges but $\sum a_n$ converges, then $\sum a_n$ is *Conditionally Convergent*
- if $\left|\sum a_n\right|$ converges, then $\sum a_n$ converges.

Direct Comparison Test:

If the series looks like another series $\sum b_n$, then:

- If $a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges also.
- If $a_n \ge b_n$ and $\sum b_n$ diverges then $\sum a_n$ diverges also.

You need to know if $\sum b_n$ converges or diverges, so you usually use a geometric series, p-series, or integrable series for the comparison. You must verify that for sufficiently large values of n, the rule of sequence of one is greater than or equal to the other term for term. Use this test when the rule of sequence if VERY SIMILAR to a known series.

Ex) compare $\frac{n}{2^n}$ to $\frac{1}{2^n}$, $\frac{1}{n^3+1}$ to $\frac{1}{n^3}$, $\frac{n^2}{\left(n^2+3\right)^2}$ to $\frac{n}{\left(n^2+3\right)^2}$

Limit Comparison Test:

(may be used instead of Direct Comparison Test most of the time)

If a_n, $b_n > 0$ and $\lim_{x\to\infty}\left|\frac{a_n}{b_n}\right|$ or $\lim_{x\to\infty}\left|\frac{b_n}{a_n}\right|$ equal any finite number, then either both $\sum a_n$ and $\sum b_n$ converge or diverge.

Use this test when you cannot compare term by term because the rule of sequence is "too UGLY" but you can still find a known series to compare with it.

Ex) compare: $\frac{3n^2+2n-1}{4n^5-6x+7}$ to $\frac{1}{n^3}$ (you can disregard the leading coefficient and all non-leading terms, looking only at the condensed degree of the leading terms: $\frac{n^2}{n^5} = \frac{1}{n^3}$.

Ratio Test:

If $a_n > 0$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = N$ (where N is a real number), then

1. $\sum a_n$ converges absolutely (and hence converges) if $N < 1$
2. $\sum a_n$ diverges if $N > 1$ or $N = \infty$
3. The test is inconclusive if $N = 1$ (use another test)

Use this test for series whose terms converge rapidly, for instance those involving exponentials and/or factorials!!!!!!

Root Test:

If $\sum a_n$ is a series with non-zero terms and $\lim_{n\to\infty}\sqrt[n]{|a_n|} = N$ (where N is a real number), then

1. $\sum a_n$ converges absolutely (and hence converges) if $N < 1$
2. $\sum a_n$ diverges if $N > 1$ or $N = \infty$
3. The test is inconclusive if $N = 1$ (use another test)

Use this test for series involving nth powers. Ex) $\sum \frac{e^{2n}}{n^n}$

Remember, if you are asked to find the ACTUAL sum of an infinite series, it must either be a Geometric series $\left(S = \frac{a_1}{1-r}\right)$ or a Telescoping Series (requires expanding and canceling terms). The telescoping series can be quite overt, such as $\sum\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$ or in "disguise" as $\sum \frac{2}{4n^2-1}$, in which case partial fraction decomposition must be used. Also note that it is possible to tell that this last series converges by Comparison tests, but the actual sum can only be given by expanding!

The only other tests that allows us to *approximate* the infinite sum are the Integral test and the Alternate Series Test. We can find the nth partial Sum S_n for any series.

$\sum_{n=1}^{\infty} a_n$ Convergence / Divergence Flow Chart for 10.1 through 10.9

Nth Term Test for Divergence

Does $\lim_{n\to\infty} a_n = 0$?
- No → $\sum_{n=1}^{\infty} a_n$ Diverges
- Yes → p-series

p-series

Does $a_n = \frac{1}{n^p}, n \geq 1$?
- Yes → Is $p > 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges
- No → Geometric Series

Geometric Series

Does $a_n = ar^{n-1}, n \geq 1$?
- Yes → Is $|r| < 1$?
  - Yes → $\sum_{n=k}^{\infty} a_n = \frac{ar^k}{1-r}$
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges
- No → Alternating Series

Alternating Series

Does $a_n = (-1)^n b_n$, or $a_n = (-1)^{n+1} b_n, b_n \geq 0$?
- Yes → Is $\lim_{n\to\infty} b_n = 0$ and $\frac{b_{n+1}}{b_n} < 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
- No → Try one of the following:

Try one of the following:

Comparison Test

Pick b_n to compare to a_n. Does $\sum b_n$ converge?
- Yes → Is $0 \leq a_n \leq b_n$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → No conclusion can be made.
- No → Is $0 \leq b_n \leq a_n$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Diverges

Limit Comparison Test

Pick b_n. Does $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c > 0$ and finite, and $a_n, b_n > 0$?
- Yes → Does $\sum_{n=1}^{\infty} b_n$ converge?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

Integral Test

Is $a_n = f(n)$ continuous, positive and decreasing on $[a, \infty)$?
- Yes → Does $\int_a^{\infty} f(n)\, dn$ converge?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Converges
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

Ratio Test

Is $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \neq 1$?
- Yes → Is $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$?
  - Yes → $\sum_{n=1}^{\infty} a_n$ Abs. Conv.
  - No → $\sum_{n=1}^{\infty} a_n$ Diverges

ROOT TEST

Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} \neq 1$?
- YES → Is $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$?
  - YES → $\sum a_n$ Abs. Conv.
  - NO → $\sum a_n$ Diverges