Related Rates Notes 2 - Similar Triangles and Shadow Problems Key

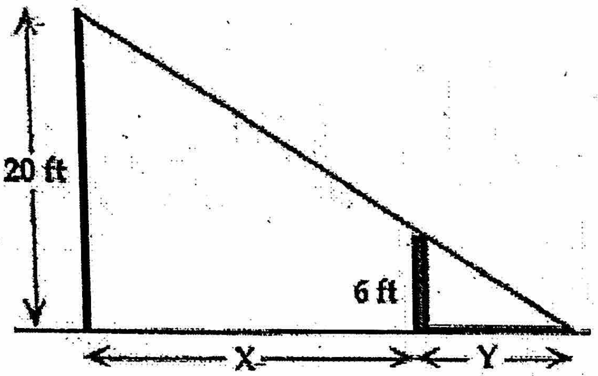
Example 1:

A man who is 6 feet tall is walking away from a lamp post at a rate of 5 feet per ~~minute~~ second.

The lamp post is 20 feet tall. The person casts a shadow on the ground in front of them. $\frac{dy}{dt}$

a) How fast is the shadow growing when the person is 30 feet from the lamp post? $\rightarrow \frac{dy}{dt}$

b) How fast is the tip of the shadow moving when the person is 30 ft from the lamp post? $\rightarrow \frac{dx}{dt} + \frac{dy}{dt}$



Notes:

1) $\frac{dx}{dt} = \textit{rate of person walking}$

2) $\frac{dy}{dt} = \textit{rate of change of shadow length}$

3) $\frac{dx}{dt} + \frac{dy}{dt} = \textit{rate of change of tip of shadow}$

$\frac{dx}{dt} = 5 \text{ ft/s}$

$x = 30 \text{ ft}$

$\frac{6}{20} = \frac{y}{x+y}$

$6(x+y) = 20y$

$6x + 6y = 20y$

$6x = 14y$

$6\left(\frac{dx}{dt}\right) = 14\left(\frac{dy}{dt}\right)$

$6(5) = 14\left(\frac{dy}{dt}\right)$

$\frac{30}{14} = \frac{dy}{dt}$

$\frac{15}{7} = \frac{dy}{dt}$

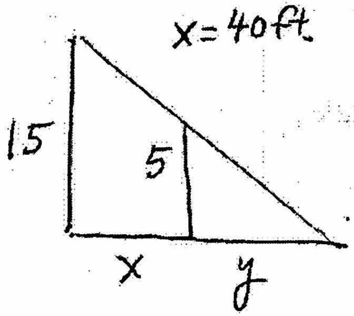
a) $\frac{dy}{dt} = \boxed{\frac{15}{7} \text{ ft/min}}$

b) $\frac{dx}{dt} + \frac{dy}{dt} = 5 + \frac{15}{7}$

$= \boxed{\frac{50}{7} \text{ or } 7.14 \text{ ft/min}}$

2. A street light is mounted at the top of a 15 ft pole. A man 5 ft tall walks towards the pole at a rate of 5ft per second. A) How fast is the tip of his shadow moving when he is 40 ft from the pole? B) How fast is the length of the shadow changing when he is 40 ft from the pole?

$x = 40 \text{ ft}$



$\frac{5}{15} = \frac{y}{x+y}$

$\frac{dx}{dt} = -5 \text{ ft/s}$

$\frac{1}{3} = \frac{y}{x+y}$

$3y = x + y$

$2y = x$

$2\left(\frac{dy}{dt}\right) = \frac{dx}{dt}$

$2\left(\frac{dy}{dt}\right) = -5$

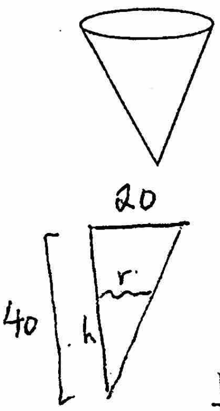
$\frac{dy}{dt} = \frac{-5}{2} \text{ ft/s}$

a) $\frac{dx}{dt} + \frac{dy}{dt} = -5 - \frac{5}{2}$

$= \boxed{-7.5 \text{ ft/s}}$

b) $\frac{dy}{dt} = \boxed{\frac{-5}{2} \text{ ft/s or } -2.5 \text{ ft/s}}$

3. A conical tank (vertex down) is 40 feet across the top and 40 feet deep. If water is leaking out of the tank at a rate of 80 cubic feet per minute, find the rate of change of the radius of the water when the water is 8 feet deep. ($V = \frac{1}{3}\pi r^2 h$)



$\frac{dV}{dt} = -80\ ft^3/min$

$h = 8\ ft$

$\frac{dr}{dt} = \_\_\_$

$\frac{r}{20} = \frac{h}{40}$

$20h = 40r$

$h = \frac{40}{20} r$

$h = 2r$, $\underline{r=4}$

* Since $h = 2r$ and $h=8$, $\underline{r=4}$

$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} r^2 (2r)$

$V = \frac{2\pi}{3} r^3$

$\frac{dV}{dt} = \frac{2\pi}{3} \cdot 3r^2 \left(\frac{dr}{dt}\right)$

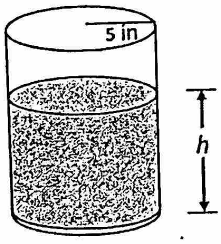
$\frac{dV}{dt} = 2\pi r^2 \left(\frac{dr}{dt}\right)$

$-80 = 2\pi (4)^2 \left(\frac{dr}{dt}\right)$

$\frac{-80}{32\pi} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = -\frac{5}{2\pi}\ ft/min}$

4. **2003 AB problem #5**



A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let *h* be the depth of the coffee in the pot, measured in inches, where *h* is a function of time, *t*, measured in seconds. The volume, *V*, of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume of a cylinder with radius *r* and height *h* is $V = \pi r^2 h$.) Find $\frac{dh}{dt}$ as a function of *h*. (This means your answer will contain the variable *h*)

$\frac{dV}{dt} = -5\pi\sqrt{h}$

$r = 5\ in.$

$V = \pi r^2 h$

$V = \pi(5)^2 h$

$V = 25\pi h$

$\frac{dV}{dt} = 25\pi\left(\frac{dh}{dt}\right)$

$-5\pi\sqrt{h} = 25\pi \frac{dh}{dt}$

$\frac{-5\pi\sqrt{h}}{25\pi} = \frac{dh}{dt}$

$\boxed{\frac{dh}{dt} = \frac{-\sqrt{h}}{5}\ in/sec.}$

AP Calculus AB Related Rates WS #4 Name Key Date

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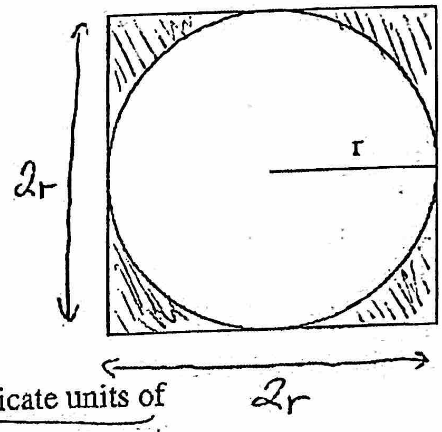
1994 AB5, BC2

1) A circle is inscribed in a square as shown in the figure. The circumference of the circle is increasing at a constant rate of 6 inches per minute. As the circle expands, the square expands to maintain the condition of tangency. (Note: A circle with radius r has circumference $C = 2\pi r$ and area $A = \pi r^2$.)



$\frac{dC}{dt} = 6 \text{ in/min}$ $\quad \frac{dC}{dt} = 2\pi\left(\frac{dr}{dt}\right)$

a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

$P = 8r$

$\frac{dP}{dt} = 8\frac{dr}{dt}$

$6 = 2\pi\left(\frac{dr}{dt}\right)$

$\frac{6}{2\pi} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{\pi} \text{ in/min}$

$\frac{dP}{dt} = 8\left(\frac{3}{\pi}\right) = \boxed{\frac{24}{\pi} \text{ in/min}}$

A_e = Area enclosed

A_c = Area circle

A_s = Area square

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.

$A = 25\pi$

$\frac{dA_e}{dt} = \_\_\_$

$A = \pi r^2$

$25\pi = \pi r^2$

$25 = r^2$

$\boxed{5 = r}$

$A_e = A_s - A_c$

$A_s = (2r)^2$

$A_c = \pi r^2$

$A_e = 4r^2 - \pi r^2$

$\frac{dA_e}{dt} = 8r\left(\frac{dr}{dt}\right) - 2\pi r\left(\frac{dr}{dt}\right)$

$\frac{dr}{dt} = \frac{3}{\pi} \text{ in/min}$

$\frac{dA}{dt} = 8(5)\left(\frac{3}{\pi}\right) - 2\pi(5)\left(\frac{3}{\pi}\right)$

$\boxed{\frac{dA}{dt} = \left(\frac{120}{\pi} - 30\right) \text{in}^2/\text{min}}$

$V = 4(t)^{1/2}$

2. Suppose that a spherical balloon grows in such a way that after t seconds, $V = 4\sqrt{t}$ in^3. How fast is the radius changing after 64 seconds? $(V = \frac{4}{3}\pi r^3)$ $\longrightarrow$ $32 = \frac{4}{3}\pi(r^3)$ $\quad \frac{24}{\pi} = r^3$

$r = \sqrt[3]{24/\pi}$

$t = 64$

$V = 4\sqrt{64} = 4 \cdot 8 = 32 \text{ in}^3$

$\frac{dV}{dt} = 4 \cdot \frac{1}{2} t^{-1/2}\left(\cancel{\frac{dt}{dt}}\right)$

$\frac{dV}{dt} = \frac{2}{\sqrt{t}} = \frac{2}{\sqrt{64}} = \frac{2}{8} = \frac{1}{4} \text{ in}^3/\text{s}$

Find $\frac{dr}{dt} = \_\_\_$

$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 \cdot \frac{dr}{dt}$

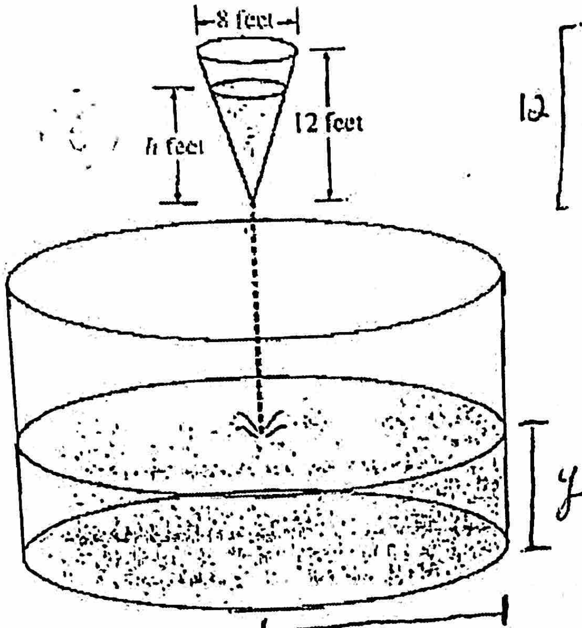
$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$

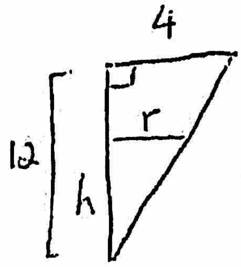
$\frac{1}{4} = 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2 \frac{dr}{dt}$

$\frac{1}{4 \cdot 4\pi\left(\sqrt[3]{\frac{24}{\pi}}\right)^2} = \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{194.974} \approx \boxed{0.005 \text{ in/s}}$

3. 1995 AB 5





$\frac{r}{4} = \frac{h}{12}$

$12r = 4h$

$r = \frac{4}{12}h$

$\boxed{r = \frac{h}{3}}$

$\frac{dh}{dt} = h - 12 \text{ ft/min}$

As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

$V = \frac{\pi}{3} r^2 h$

$V = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h$

$V = \frac{\pi}{3} \cdot \frac{h^2}{9} \cdot h$

$\boxed{V = \frac{\pi}{27} h^3}$

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

$V = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} \cdot 3h^2 \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9} h^2 \cdot \left(\frac{dh}{dt}\right)$

$\frac{dV}{dt} = \frac{\pi}{9} h^2 \cdot (h-12)$

$\frac{dV}{dt} = \frac{\pi}{9}(3)^2(3-12)$

$= \frac{\pi}{\cancel{9}} \cdot \cancel{9}(-9)$

$\boxed{\frac{dV}{dt} = -9\pi \text{ ft}^3/\text{min}}$

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure. $(V = \pi a^2 y)$

$\frac{dV}{dt} = 9\pi \text{ ft}^3/\text{min}$

Area(base) $= 400\pi$

$A = \pi r^2$

$400\pi = \pi r^2$

$400 = r^2$

$20 = r$

$r = 20 \text{ ft}$

$a = 20$

$V = \pi (20)^2 y$

$V = 400\pi y$

$\frac{dV}{dt} = 400\pi \left(\frac{dy}{dt}\right)$

$9\pi = 400\pi \left(\frac{dy}{dt}\right)$

$\frac{9}{400} = \frac{dy}{dt}$

$\boxed{\frac{dy}{dt} = \frac{9}{400} \text{ ft/min}}$