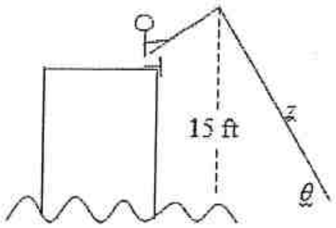
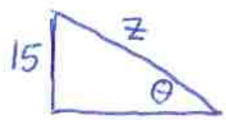
Calculus Related Rates involving Trig - Day 3 Notes

Key

Example 1: A fish is reeled in at a rate of 1 foot per second from a point 15 feet above the water (see figure below). At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?





$\frac{dz}{dt} = -1 \text{ ft/sec}$

$z = 25 \text{ ft.}$

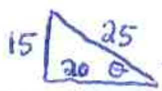
Find $\frac{d\theta}{dt}$

$\sin\theta = \frac{15}{z}$

$\sin\theta = 15z^{-1}$

$\cos\theta\left(\frac{d\theta}{dt}\right) = -15z^{-2}\left(\frac{dz}{dt}\right)$

$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{-15}{z^2}\left(\frac{dz}{dt}\right)$



$\cos\theta = \frac{20}{25} = \frac{4}{5}$

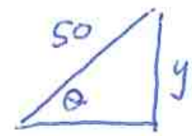
$\left(\frac{4}{5}\right)\left(\frac{d\theta}{dt}\right) = \frac{-15}{25^2}\cdot(-1)$

$\frac{d\theta}{dt} = \frac{15}{25^2}\cdot\frac{5}{4}$

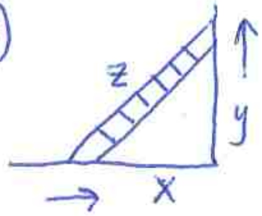
$\boxed{\frac{d\theta}{dt} = \frac{3}{100} \text{ rad/sec}}$

2) A ladder, 50 ft long, is being pushed against the wall at a rate of 5 ft / sec. When the bottom of the ladder is 30 ft from the wall:

a) What is the velocity at the top of the ladder?
b) At what rate is the area of the triangle enclosed by the ladder, wall, and floor changing?
c) At what rate is the angle formed by the ladder and the floor changing at that time?



a)



$x^2 + y^2 = z^2$

$2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right) = 2z\left(\frac{dz}{dt}\right)$

$x = 30 \quad dx/dt = -5$

$y = \underline{40} \quad dy/dt = \_\_\_$

$z = 50 \quad dz/dt = 0$

$2(30)(-5) + 2(40)\left(\frac{dy}{dt}\right) = 0$

$\boxed{\frac{dy}{dt} = 3.75 \text{ ft/sec}}$

b) $A = \underbrace{\tfrac{1}{2}x}_{f}\,\underbrace{y}_{g}$ *product Rule

$\frac{dA}{dt} = \underbrace{\frac{1}{2}\left(\frac{dx}{dt}\right)}_{f'}\cdot \underbrace{y}_{g} + \underbrace{\frac{1}{2}x}_{f}\cdot\underbrace{\frac{dy}{dt}}_{g'}$

$\frac{dA}{dt} = \frac{1}{2}(-5)(40) + \frac{1}{2}(30)(3.75)$

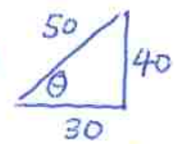
$\frac{dA}{dt} = -100 + 56.25$

$\boxed{\frac{dA}{dt} = -43.75 \text{ ft}^2\text{/sec}}$

c) $\sin\theta = \frac{y}{50}$

$\sin\theta = \frac{1}{50}y$

$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dy}{dt}\right)$



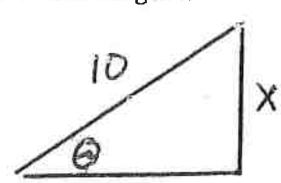
$\cos\theta = \frac{30}{50} = \frac{3}{5}$

$\left(\frac{3}{5}\right)\frac{d\theta}{dt} = \frac{1}{50}(3.75)$

$\frac{d\theta}{dt} = \frac{3.75}{50}\cdot\frac{5}{3}$

$\boxed{\frac{d\theta}{dt} = 0.125 \text{ or } \frac{1}{8} \text{ rad/sec}}$

3) The angle θ is increasing at a constant rate of 6 radians per hour. At what rate is the side of length x increasing when x = 6 feet?



$\frac{d\theta}{dt} = 6 \text{ rad/hr}$

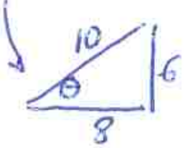
$\frac{dx}{dt} = \_\_\_?$

$x = 6$

$\sin\theta = \frac{x}{10}$

$\sin\theta = \frac{1}{10}x$

$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{10}\left(\frac{dx}{dt}\right)$



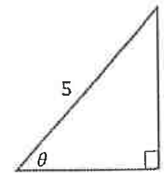
$\cos\theta = \frac{8}{10} = \frac{4}{5}$

$\left(\frac{4}{5}\right)(6) = \frac{1}{10}\left(\frac{dx}{dt}\right)$

$\frac{24}{5}\cdot\frac{10}{1} = \frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = 48 \text{ ft/hr}}$

4) In the right triangle shown, the angle θ is increasing at a constant rate of 2 radians per hour. At what rate is the side length x increasing when x = 4 feet?



$\sin\theta = \frac{x}{5}$

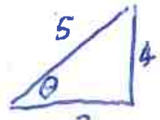
$\sin\theta = \frac{1}{5}x$

$\cos\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{5}\left(\frac{dx}{dt}\right)$

$\frac{d\theta}{dt} = 2 \text{ rad/hr}$

$\frac{dx}{dt} = \_\_?$

$x = 4 \text{ ft}$



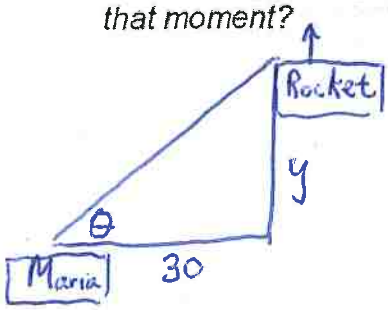
$\cos\theta = \frac{3}{5}$

$\left(\frac{3}{5}\right)(2) = \frac{1}{5}\left(\frac{dx}{dt}\right)$

$\frac{6}{5} \cdot \frac{5}{1} = \frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = 6 \text{ ft/hr}}$

5) A model rocket is launched 30 feet from Maria, and is rising vertically at a constant rate of 20 ft/s when the rocket has an elevation of 40 feet. How fast is the angle of elevation from Maria to the rocket changing at that moment?



$\tan\theta = \frac{y}{30}$

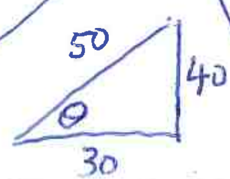
$\tan\theta = \frac{1}{30}y$

$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{30}\left(\frac{dy}{dt}\right)$

$\frac{dy}{dt} = 20 \text{ ft/sec}$

$y = 40$

$\frac{d\theta}{dt} = \_\_?$



$\sec\theta = \frac{50}{30}$

$\sec\theta = \frac{5}{3}$

$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{30}\left(\frac{dy}{dt}\right)$

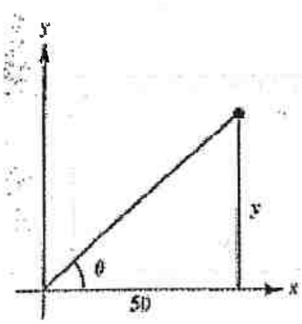
$\left(\frac{5}{3}\right)^2\left(\frac{d\theta}{dt}\right) = \frac{1}{30}(20)$

$\frac{d\theta}{dt} = \frac{9}{25} \cdot \frac{20}{30}$

$\frac{d\theta}{dt} = \frac{9}{25} \cdot \frac{2}{3}$

$\boxed{\frac{d\theta}{dt} = \frac{6}{25} \text{ rad/sec}}$

6) **Angle of Elevation** A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.



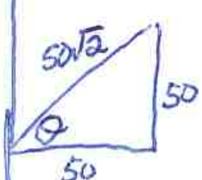
$\frac{dy}{dt} = 4 \text{ m/sec}$

$\frac{d\theta}{dt} = \_\_?$

$y = 50$

$\tan\theta = \frac{y}{50} = \frac{1}{50}(y)$

$\sec^2\theta\left(\frac{d\theta}{dt}\right) = \frac{1}{50}\left(\frac{dy}{dt}\right)$



$\sec\theta = \frac{50\sqrt{2}}{50}$

$\sec\theta = \sqrt{2}$

$(\sqrt{2})^2 \frac{d\theta}{dt} = \frac{1}{50}(4)$

$\frac{d\theta}{dt} = \frac{1}{2} \cdot \frac{4}{50}$

$\boxed{\frac{d\theta}{dt} = \frac{1}{25} \text{ rad/sec}}$