Key

AP Calculus – 4.4 Notes - L'Hopital's Rule and Indeterminate Form

Recall: When evaluating limits, first try direct substitution! $\lim_{x\to 3} \frac{2x-5}{x} = \frac{6-5}{3} = \frac{1}{3}$

1. $\lim_{x\to 2} \frac{x^2-7x+10}{x-2} = \frac{4-14+10}{2-2} \to \frac{0}{0} \to \lim_{x\to 2} \frac{(x-5)\cancel{(x-2)}}{\cancel{(x-2)}} \to 2-5 = \boxed{-3}$

$\lim_{x\to 2} \frac{2x-7}{1} \to 4-7 = \boxed{-3}$

Indeterminate Form

L'Hospital's Rule:

Suppose $f(a) = 0$ and $g(a) = 0$ and $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$. L'Hopital's Rule allows you to apply the following:

$$\lim_{x\to a} \frac{f(x)}{g(x)} \to \frac{f'(a)}{g'(a)}$$

Evaluate each limit. Use L'Hospital's when possible.

2. $\lim_{x\to 2} \frac{x-2}{3x^3-6x^2+x-2} \to \frac{0}{0}$

$\lim_{x\to 2} \frac{1}{9x^2-12x+1} \to \frac{1}{9(4)-24+1} \to \frac{1}{36-24+1} \to \boxed{\frac{1}{13}}$

3. $\lim_{x\to 0} \frac{\sin(6x)}{x} \to \frac{0}{0}$

$\lim_{x\to 0} \frac{\cos(6x)\cdot 6}{1} \to \frac{6\cos(0)}{1} \to 6(1) = \boxed{6}$

4. $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} \to \frac{0}{0}$

$\lim_{x\to 0} \frac{-(-\sin x)}{2x} \to \frac{\sin 0}{2(0)} \to \frac{0}{0}$

$\lim_{x\to 0} \frac{\cos x}{2} \to \frac{\cos 0}{2} = \boxed{\frac{1}{2}}$

5. $\lim_{x\to \infty} \frac{2x^2}{e^{2x}} \to \frac{\infty}{\infty}$

$\lim_{x\to \infty} \frac{4x}{2e^{2x}} \to \frac{\infty}{\infty}$

$\lim_{x\to \infty} \frac{4}{4e^{2x}} \to \frac{0}{\infty} = \boxed{0}$

* OR Apply comparative growth rate

L<R<P<E

$\lim_{x\to \infty} \frac{\text{slower}}{\text{faster}} = \boxed{0}$

L'HOSPITAL'S IS NOT THE QUOTIENT RULE!!

6. $\frac{d}{dx} \frac{\sin(6x)}{x}$

$$f'(x) = \frac{\overbrace{\cos(6x)\cdot 6}^{f'}\overbrace{x}^{g} - \overbrace{\sin(6x)}^{f}\cdot\overbrace{1}^{g'}}{\underbrace{x^2}_{g^2}} \to \boxed{\frac{6x\cos(6x)-\sin(6x)}{x^2}}$$

Practice Problems:

Find the following. Use L'Hôpital's when possible.

1. $\lim_{x\to1}\frac{x-1}{x^2-3x+2} \to \frac{0}{0}$ $\lim_{x\to1}\frac{1}{2x-3} = \frac{1}{-1} = \boxed{-1}$	2. $\lim_{x\to-5}\frac{x^2-2x-35}{x+5} \to \frac{0}{0}$ $\lim_{x\to-5}\frac{2x-2}{1} \to \boxed{-12}$ $-5(2)-2$	3. $\lim_{x\to0}\frac{4x}{\ln(x+1)} \to \frac{0}{0}$ $\lim_{x\to0}\frac{4}{\frac{1}{x+1}} \to \frac{4}{\frac{1}{1}} = \boxed{4}$
4. $\lim_{x\to0}\frac{x-1}{x^2-3x+2} \to \boxed{\frac{-1}{2}}$	5. $\lim_{x\to1}\frac{2(x^2-1)}{\ln x^2} \to \frac{0}{0}$ $\lim_{x\to1}\frac{2(2x)}{2(\frac{1}{x})} \to \frac{4}{2} = \boxed{2}$	6. $\frac{d}{dx}\frac{6x^2+x}{\sin(x)}$ Quotient Rule $\frac{(12x+1)\sin x - (6x^2+x)\cos x}{\sin^2 x}$

16. If $f(x) = 2x^3 + 5$, then $\lim_{x\to0}\frac{f(x)-f(0)}{x^3}$ is $\to \lim_{x\to0}\frac{2x^3+5-5}{x^3} \to \lim_{x\to0}\frac{2x^3}{x^3} \to \frac{0}{0}$

L'Hôpital's $\to \lim_{x\to0}\frac{f'(x)}{3x^2} \to \frac{6x^2}{3x^2} \to \boxed{2}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) The limit does not exist.

17. Functions f, g, and h are twice-differentiable functions with $g(3) = h(3) = 5$. The line $y = 5 + \frac{1}{2}(x-3)$ is tangent to both the graph of g at $x = 3$ and the graph of h at $x = 3$.

a. Find $h'(3)$.

$h(x)$ is tangent to $y = \frac{1}{2}(x-3)+5$

* shares same slope, so $h'(3) = \frac{1}{2}$

b. Let a be the function given by $a(x) = \underbrace{2x^3}_{f}\underbrace{h(x)}_{g}$. Write an expression for $a'(x)$. Find $a'(3)$.

$$a'(x) = \overbrace{6x^2}^{f'} \cdot \overbrace{h(x)}^{g} + \overbrace{2x^3}^{f} \cdot \overbrace{h'(x)}^{g'}$$

$$a'(3) = 6(3)^2 \cdot h(3) + 2(3)^3 \cdot h'(3)$$

$$a'(3) = 6(9)(5) + 2(27)\left(\tfrac{1}{2}\right)$$

$$\boxed{a'(3) = 297}$$

c. The function h satisfies $h(x) = \frac{x^2-9}{1-(f(x))^3}$ for $x \neq 3$. It is known that $\lim_{x\to 3} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x\to 3} h(x) = 5$ to find $f(3)$ and $f'(3)$. Show the work that leads to your answers.

$$\lim_{x\to 3} \frac{x^2-9}{1-[f(x)]^3} \to \frac{0}{0} \to \text{L'Hospital's} \to \lim_{x\to 3} \frac{2x}{-3[f(x)]^2 \cdot f'(x)}$$

$1-[f(3)]^3 = 0$

$(f(3))^3 = 1$

$f(3) = 1$

Chain Rule

out: $-[\ ]^3$

in: $f(x)$

$$\lim_{x\to 3} \frac{2(3)}{-3[f(3)]^2 f'(3)} = 5$$

$$\frac{6}{-3[1]^2 \cdot f'(3)} = \frac{5}{1}$$

$$-3 \cdot 5 \cdot f'(3) = 6$$

$$f'(3) = \frac{6}{-15}$$

$$\boxed{f'(3) = \frac{-2}{5}}$$