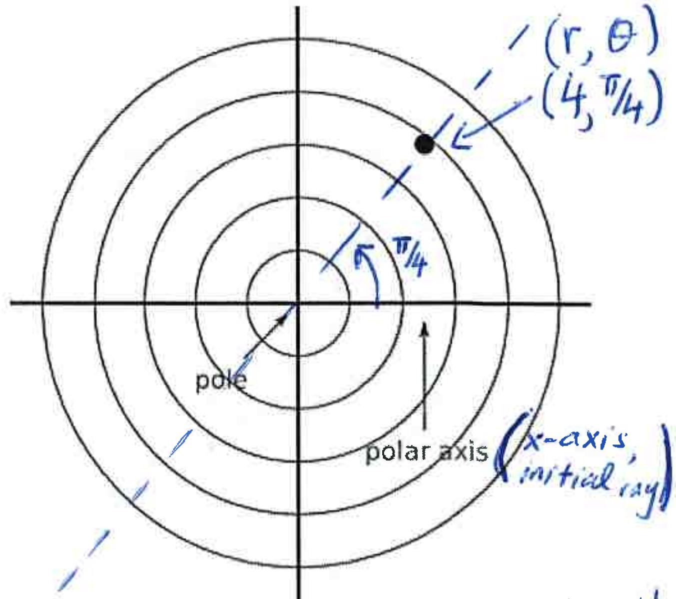
Key

BC Calculus – 9.4-prep Notes – Graphing Polar Equations

A rectangular coordinate system is only one way to navigate through a Euclidean plane. Such coordinates, (x, y), known as **rectangular coordinates**, are useful for expressing functions of y in terms of x. Curves that are not functions are often more easily expressing in an alternative coordinate system called **polar coordinates**.

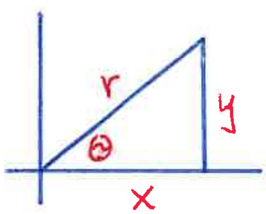


In a polar coordinate system, we still have the traditional x- and y- axes. The intersection of these axes, the old origin, is called the **pole**. Similar to navigating on the Unit Circle, we can now get to any point in 2-D space by specifying an independent choice of an angle, θ, from the initial ray, **polar axis**, then walking out along that terminal ray a specified amount, r, in either direction.

Although the angle is the independent variable, we express the point in the polar plane as (r, θ). The point to the right would have coordinates of $\left(4, \frac{\pi}{4}\right)$.

walk 4 units out along the angle $\pi/4$ from the polar axis

Example 1:
Find several other equivalent polar coordinates for the point shown above, then find the equivalent rectangular coordinate.



$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$

Equivalent polar coordinates to $(4, \pi/4)$
(r, θ)

i) $(-4, 5\pi/4)$ * $\frac{\pi}{4} + \frac{4\pi}{4} = \frac{5\pi}{4}$

ii) $(4, -7\pi/4)$ * $\frac{\pi}{4} - \frac{8\pi}{4} = -\frac{7\pi}{4}$

iii) $(-4, -3\pi/4)$ * $\frac{\pi}{4} - \frac{4\pi}{4} = -\frac{3\pi}{4}$

equivalent rectangular coordinate

$x = r\cos\theta \rightarrow x = 4\cos(\pi/4) \rightarrow 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = r\sin\theta \rightarrow y = 4\sin(\pi/4) \rightarrow 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$\boxed{(x, y) = (2\sqrt{2}, 2\sqrt{2})}$

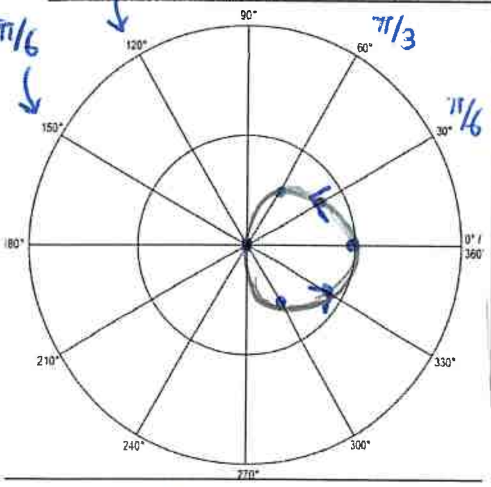
Why use polar coordinates? Graphs that aren't functions in rectangular form $f(x)$ can still be functions in polar form $r(\theta)$. Some of these curves can be quite elaborate and are more easily expressed as polar, rather than rectangular equations, as the following calculator exploration will demonstrate.

* Infinite representations of $(4, \pi/4)$ in polar form

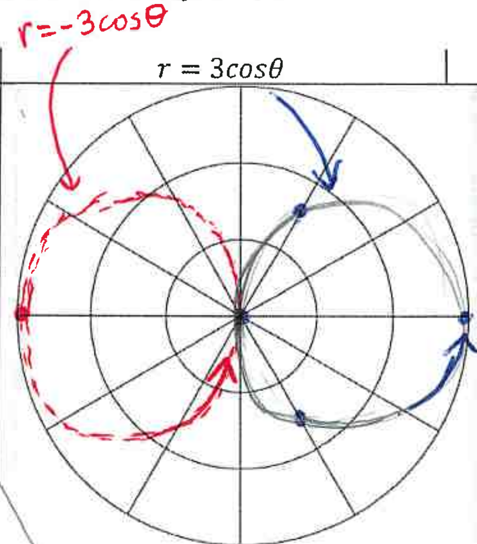
Example 2: *Graphing Calculator : Mode → $\boxed{\text{Polar}}$ (POL)

Put your graphing calculator in POLAR mode and RADIAN mode. Graph the following equations on your calculator, sketch the graphs on this sheet, and answer the questions.

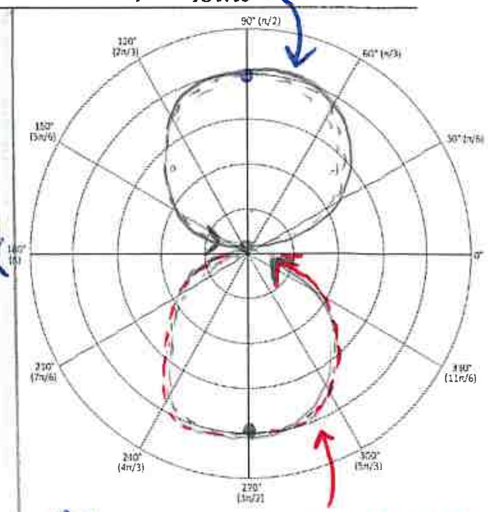
$r = \cos\theta$



$r = 3\cos\theta$



$r = 4\sin\theta$



θ	r
0	1
π/6	√3/2
π/3	1/2

θ	r
π/2	0
2π/3	-1/2
5π/6	-√3/2
π	

*Traces in CCW and 1 rotation in [0,π]

θ	r
0	3
π/2	0
π	-3

θ	r
0	0
π/2	4
π	0

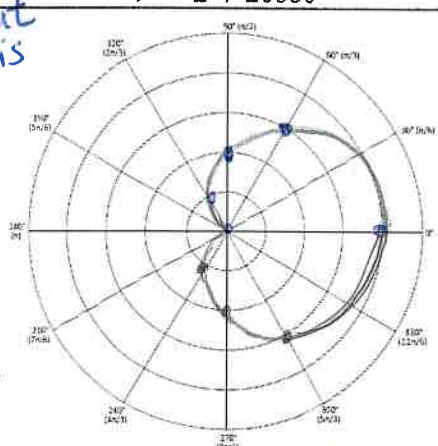
What do you notice about the above graphs?

1) diameter of circle matches the coefficient
2) traces out in CCW, 1 rotation [0,π]
3) cosine graph (symmetry about the x-axis), sine graph (symmetry about y-axis)

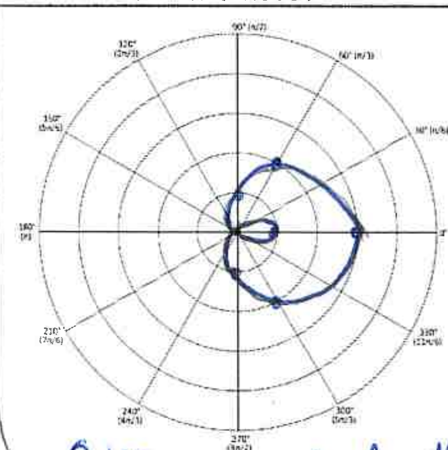
Example 3a:

*symmetric about x-axis

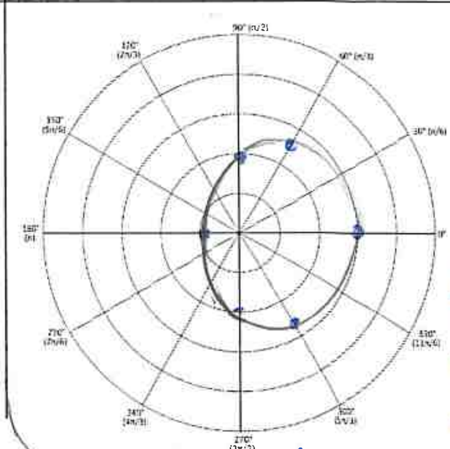
$r = 2 + 2\cos\theta$



$r = 1 + 2\cos\theta$



$r = 2 + \cos\theta$



θ	r
0	4
π/3	3
π/2	2
2π/3	1
$\boxed{\pi}$	$\boxed{0}$

polar zero (origin) at θ=π

cardioid [0, 2π]

*2+2=4 (outer diameter)
*2-2=0 (inner diameter)

(constant ± coefficient)

θ	r
0	3
π/3	2
π/2	1
$\boxed{2\pi/3}$	$\boxed{0}$
π	-1
$\boxed{4\pi/3}$	$\boxed{0}$

2 polar zeros at $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

*cardioid with inner loop [0, 2π] (limaçon)

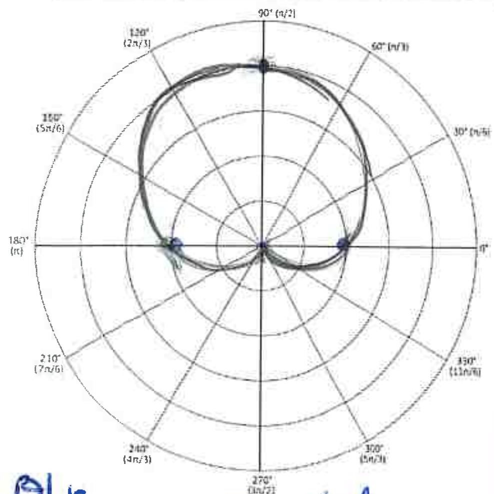
*1+2=3 (outer diameter)
1-2=-1 (inner loop diameter)

*dimpled limaçon
*No polar zeros [0, 2π]

θ	r
0	3
π/3	2.5
π/2	2
π	1

Example 3b: *symmetric about y-axis

$r = 2 + 2sin\theta$

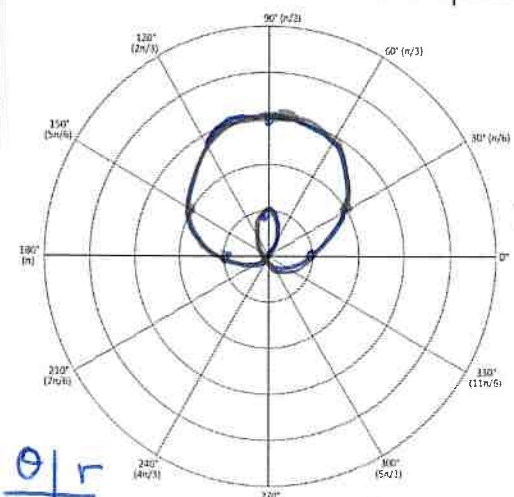


θ	r
0	2
$\pi/2$	4
π	2
$\boxed{3\pi/2}$	$\boxed{0}$
2π	2

polar zero at $\theta = \frac{3\pi}{2}$

*Inverted Cardioid

$r = 1 + 2sin\theta$



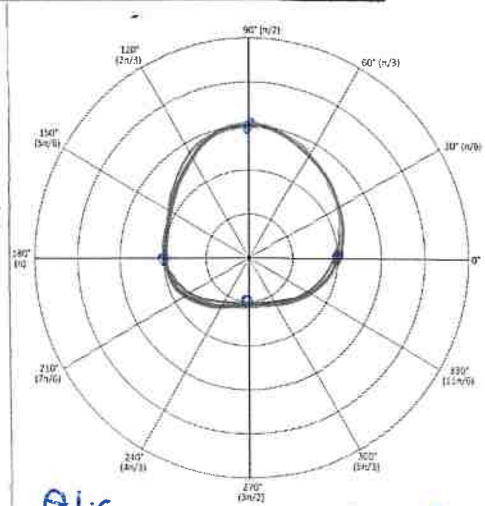
θ	r
0	1
$\pi/6$	2
$\pi/2$	3
π	1

θ	r
$\boxed{7\pi/6}$	$\boxed{0}$
$3\pi/2$	-1
$\boxed{11\pi/6}$	$\boxed{0}$
2π	1

2 polar zeros

*limaçon $[0, 2\pi]$

$r = 2 + sin\theta$



θ	r
0	2
$\pi/2$	3
π	2

θ	r
$\frac{3\pi}{2}$	1
2π	2

*dimpled limaçon $[0, 2\pi]$

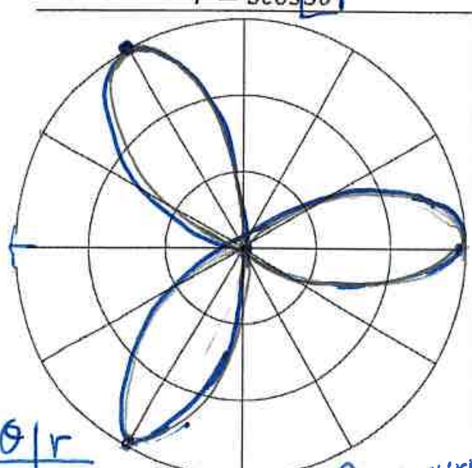
Which graphs go through the pole? cardioid (once), limaçon (twice), dimpled limaçon (none)
Which ones do not go through the pole? dimpled limaçon
Which ones have an inner loop? limaçon

Example 4:

petal length

odd number indicates the number of petals

$r = 3cos\boxed{3}\theta$



θ	r
0	3
$\boxed{\pi/6}$	$\boxed{0}$
$\pi/3$	-3
$\boxed{\pi/2}$	$\boxed{0}$

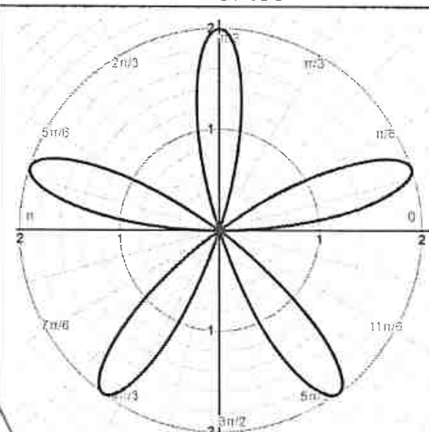
θ	r
$\frac{2\pi}{3}$	3
$\boxed{5\pi/6}$	$\boxed{0}$
π	-3

*Rose curve with 3 petals $[0, \pi]$

*3 polar zeros

*x-axis symmetry

$r = 2sin5\theta$



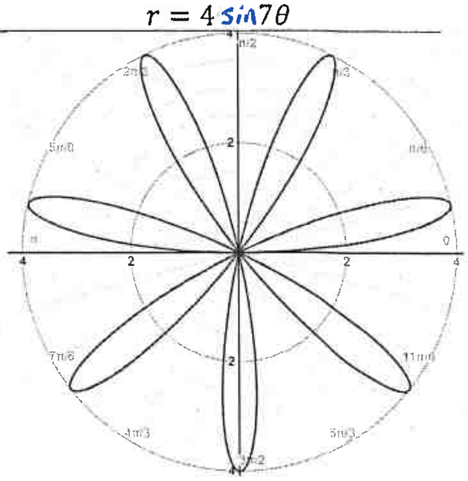
*petal length is 2

*5 polar zeros (5 petals)

*completes cycle in $[0, \pi]$

*y-axis symmetry

$r = 4\,sin7\theta$



*7 polar zeros

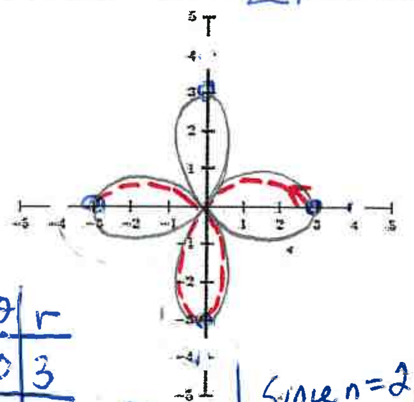
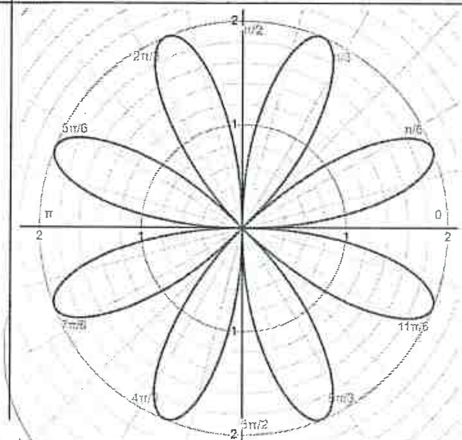
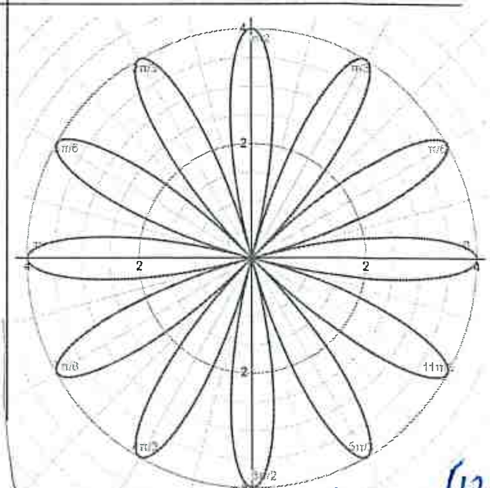
*y-axis symmetry

*petal length is 4

petal length

For $n\theta$, $2n$ petals if n is even

Example 5:

$r = 3\cos \boxed{2}\theta$	$r = 2\sin 4\theta$	$r = 4\cos 6\theta$
		

θ	r
0	3
$\boxed{\pi/4}$	$\boxed{0}$
$\pi/2$	-3
$\boxed{3\pi/4}$	$\boxed{0}$
π	3

θ	r
$\boxed{5\pi/4}$	$\boxed{0}$
$3\pi/2$	-3
$\boxed{7\pi/4}$	$\boxed{0}$
2π	3

different location from the starting point at $\theta = \pi$

Since $n=2$, $2n \to 2(2) = 4$ 4 petals

* completes cycle in $[0, 2\pi]$
* x-axis symmetry

* petal length is 2
* 4θ means 8 petals
* symmetrical about y-axis

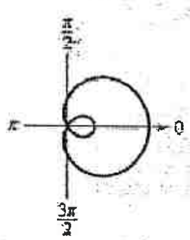
* petal length is 4
* 6θ means 12 petals (12 polar zeros)
* symmetric about x-axis

What do you notice about the above graphs?

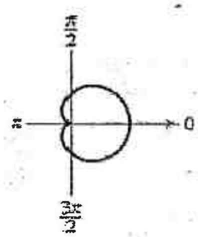
* n petals if n is odd $[0, \pi]$
* $2n$ petals if n is even $[0, 2\pi]$

Limacons

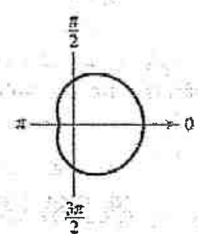
$r = a \pm b\cos\theta$
$r = a \pm b\sin\theta$
$(a > 0, b > 0)$



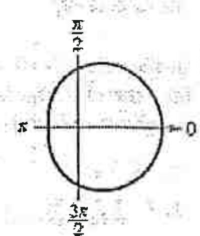
$\frac{a}{b} < 1$
Limaçon with inner loop



$\frac{a}{b} = 1$
Cardioid (heart-shaped)



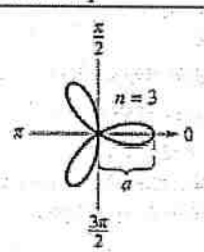
$1 < \frac{a}{b} < 2$
Dimpled limaçon



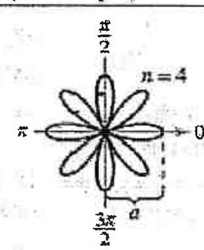
$\frac{a}{b} \geq 2$
Convex limaçon

Rose Curves

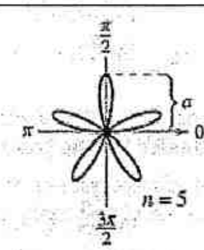
n petals if n is odd
2n petals if n is even



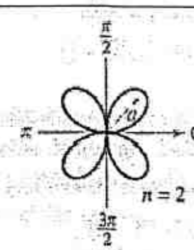
$r = a\cos n\theta$
Rose curve



$r = a\cos n\theta$
Rose curve

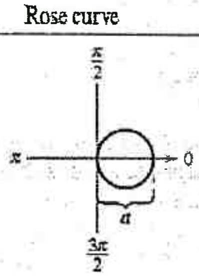


$r = a\sin n\theta$
Rose curve

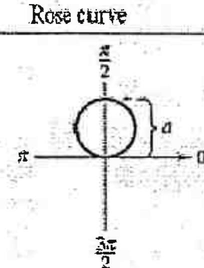


$r = a\sin n\theta$
Rose curve

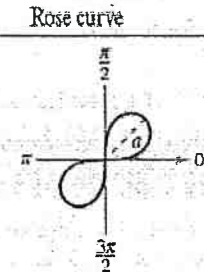
Circles and Lemniscates



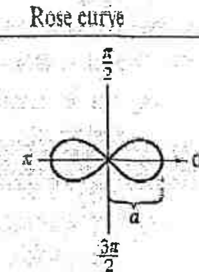
$r = a\cos\theta$
Circle



$r = a\sin\theta$
Circle



$r^2 = a^2\sin 2\theta$
Lemniscate



$r^2 = a^2\cos 2\theta$
Lemniscate