BC Calculus – 9.5a & 9.5b - Derivatives and Integrals for Vector-Valued Functions

Vector basics:

- Vectors have magnitude (length) and direction.
- Vectors can be represented by directed line segments.
- Vectors are equal if they have the same direction and magnitude.
- Magnitude is designated by $||v||$
- Vectors have a horizontal and vertical component.
- Component form of a vector is $\langle x, y \rangle$

1. Find the component form and magnitude of the vector that has an initial point of (1,2) and terminal point (5,4).

   Component form:

   Magnitude:

Vector-Valued Functions: $r(t) = \langle f(t), g(t) \rangle$ where $f(t)$ and $g(t)$ are the component functions with the parameter t.

Differentiation of Vector-Valued Functions

If $r(t) = \langle f(t), g(t) \rangle$ then

Properties of the derivative for vector-valued functions

$$\frac{d}{dt}[c \cdot r(t)] = c \cdot r'(t)$$

$$\frac{d}{dt}[r(t) \cdot s(t)] = r'(t)s(t) + r(t)s'(t)$$

$$\frac{d}{dt}[r(t) \pm s(t)] = r'(t) \pm s'(t)$$

$$\frac{d}{dt}[r(s(t))] = r'(s(t)) \cdot s'(t)$$

1. $r(t) = \langle 2t^2 + 4t + 1,\ 3t^3 - 4t \rangle$ then $r'(t) =$

2. $r(t) = \langle t^3 + 5,\ 2t \rangle$ find $\frac{d}{dt}r(2t)$

3. The path of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t^2, \sin t \rangle$. Find the slope of the path of the particle at $t = \frac{3\pi}{4}$.

(20)

9.5b - Integrals for Vector-Valued Functions

Integration of Vector-Valued Functions

If $r(t) = \langle f(t), g(t) \rangle$ then

1. Find $r(t)$ if $r'(t) = \langle 4e^{2t}, 2e^{t} \rangle$ and $r(0) = \langle 2, 0 \rangle$

2. Find $r(t)$ if $r'(t) = \langle \sec^2 t, \frac{1}{1+t^2} \rangle$

3. $\int_{-1}^{1} \langle t^3, t^{\frac{1}{5}} \rangle \, dt$

For problems 1-6, find the vector-valued function $f(t)$ that satisfies the given initial conditions.

1. $f(0) = \langle 2, 4 \rangle$, $f'(t) = \langle 2e^{t}, 3e^{3t} \rangle$

2. $f(0) = \langle \frac{1}{2}, -1 \rangle$, $f'(t) = \langle te^{-t^2}, -e^{-t} \rangle$

9.5a Derivatives of Vector-Valued Functions

Calculus

Practice

(21)

Each problem contains a vector-valued function. Find the given first or second derivative.

1. $f(t) = \langle 4t^3 + 2t^2 + 7t,\ 4t^2 + 3t \rangle$, then $f'(t) =$

2. $f(t) = \langle 3\sin 2t, 4\cos 3t \rangle$, then $f'\left(\frac{\pi}{6}\right) =$

3. $f(t) = \langle 3e^{2t},\ 5e^{4t} \rangle$, then $f''(t) =$

4. $f(t) = \langle t^{-2},\ (t+1)^{-1} \rangle$, then $f''(-2) =$

5. $f(t) = \langle e^t + e^{-t},\ e^t - e^{-t} \rangle$, then $f'(t) =$

6. $f(t) = \langle 2\sin 4t, 2\cos 3t \rangle$, then $f'(t) =$

7. $f(t) = \langle t\sin t,\ t\cos t \rangle$, then $f'\left(\frac{\pi}{2}\right) =$

8. $f(t) = \langle 3t^2 + 6t + 1,\ 4t^3 - 2t^2 + 6t \rangle$, then $f'(1) =$

22

9. The path of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t^3 + 2t^2 + t,\ 2t^3 - 4t \rangle$. Find the slope of the path of the particle at $t = 3$.

10. The position of a particle moving in the xy-plane is defined by the vector-valued function, $f(t) = \langle t^3 - 6t^2,\ 2t^3 - 9t^2 - 24t \rangle$. For what value of $t \geq 0$ is the particle at rest?

9.5a Derivatives of Vector-Valued Functions

Test Prep

11. **Calculator active.** The path of a particle moving along a path in the xy-plane is given by the vector-valued function f and f' is defined by $f'(t) = \langle t^{-1},\ 2ke^{kt} \rangle$ where k is a positive constant. The line $y = 4x + 5$ is parallel to the line tangent to the path of the particle at the point where $t = 2$. What is the value of k?

12. At time t, $0 \leq t \leq 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle t \sin t, \cos 2t \rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{2}$.

9.5b Find $f(t)$

23

3. $f(0) = \langle 3, 1\rangle$, $f'(t) = \langle 6t^2, 4t\rangle$

4. $f(0) = \langle -2, 5\rangle$, $f'(t) = \langle 2\cos t, -3\sin t\rangle$

5. $f'(0) = \langle 3, 0\rangle, f(0) = \langle 0, 3\rangle$, $f''(t) = \langle 5\cos t, -2\sin t\rangle$

6. $f'(0) = \langle 0, 2\rangle, f(0) = \langle 3, 0\rangle, f''(t) = \langle 4t^3, 3t^2\rangle$

7. **Calculator active.** For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, 4)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+3}}{e^t}$ and $\frac{dy}{dt} = \cos^2 t$. Find the x-coordinate of the particles position at time $t = 5$.

8. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 8t^3 + 2t, 10t^4\rangle$. If $f(1) = \langle 3, 7\rangle$, what is $f(-1)$?

(24)

9. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 3t^2, 3 \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

10. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 2, \frac{\cos t}{e^t} \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

9.5b Integrating Vector-Valued Functions

Test Prep

11. **Calculator active.** A remote controlled car travels on a flat surface. The car starts at the point with coordinates $(7, 6)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position change at rates given by $x'(t) = -10 \sin t^2$ and $y'(t) = 9 \cos(2 + \sqrt{t})$, where $x(t)$ and $y(t)$ are measured in feet and t is measured in minutes. Find the y-coordinate of the position of the car at time $t = 1$.

12. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 2 + 20t - 4t^3,\ 6t^2 + 2t \rangle$. If $f(1) = \langle 5, -3 \rangle$, what is $f(-1)$?