

BC Calculus – 10.5b Notes – Alternating Series Error Bound

\* Must be a converging alternating series!

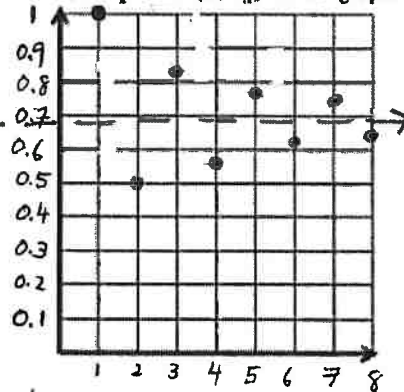
key ①

Use the alternating harmonic series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  to fill in the table below.

n		1	2	3	4	5	6	7	8
terms $a_n$	fractions	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	$-\frac{1}{6}$	$\frac{1}{7}$	$-\frac{1}{8}$
	decimals	1	-0.5	0.33	-0.25	0.2	-0.166	0.143	-0.125
partial sums $S_n$	fractions	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{47}{60}$	$\frac{37}{60}$	$\frac{319}{420}$	$\frac{533}{840}$
	decimals	1	0.5	0.833	0.583	0.783	0.616	0.759	0.634

\* This series will keep alternating and bouncing above and below the line until eventually the partial sums approach  $\ln 2$  (closer to the line)

Plot the points  $(n, S_n)$  on the graph.



\* The actual sum of this series is  $\ln 2 \approx 0.6931$

Error: $ S - S_n $	$R_1$	$R_2$						
	$ \ln 2 - 1 $	$ \ln 2 - 1/2 $	$ \ln 2 - 5/6 $	$ \ln 2 - 7/12 $	$ \ln 2 - 47/60 $			
$\leq$	$\leq a_2 (0.5)$	$\leq a_3 (0.33)$	$\leq 0.25$	$\leq 0.2$	$\leq 0.166$	$\leq 0.143$	$\leq 0.125$	

**Alternating Series Error Bound**

If you have an alternating series that converges, we can approximate the sum of the series!

$$|S - S_n| = |R_n| \leq |a_{n+1}|$$

- $S$ : Sum of the series
- $S_n$ : Partial sum
- $R_n$ : Remainder (or error)
- $R_n = S - S_n$
- $a_{n+1}$  = next term (Error Bound)

Actual Remainder many times is not known to us.

Error bound is the next term (can be found)

\* Error Bound is the boundary of how far off your actual error is.

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1. Determine the number of terms required to approximate the sum of the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  with an error less than  $10^{-3}$ .

$$|S - S_n| \leq |a_{n+1}|$$

\*what makes  $\frac{1}{n+1} < 10^{-3}$   
or  $\frac{1}{n+1} < 0.001$

$$\left. \begin{array}{l} \frac{1}{n+1} < \frac{1}{1000} \\ 1000 < n+1 \\ 999 < n \end{array} \right\} \begin{array}{l} \text{If } n > 999 \text{ terms,} \\ \text{the partial sum's approximation} \\ \text{is off by no more than} \\ \frac{1}{1000} \text{ or } 0.001 \end{array}$$

2. If the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n+2}$  is approximated by the partial sum with 10 terms, what is the alternating series error bound?

$$a_{n+1} = \left| \frac{1}{5(n+1)+2} \right| = \frac{1}{57}$$

The partial sum  $S_{10}$  is off by no more than  $\frac{1}{57}$  of the Actual sum

3. Calculator active. Approximate an interval of the sum of the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n^2}$  using the Alternating Series Error Bound for the first 5 terms.

\*Math  $\rightarrow 0$  (in calculator)

Approx:  $S_5 = \sum_{n=1}^5 a_n = 3.354$

$$3.354 - 0.111 \leq S \leq 3.354 + 0.111$$

$$\boxed{3.243 \leq S \leq 3.465}$$

$|a_6| = \left| \frac{4}{6^2} \right| = \frac{1}{9} = 0.11\bar{1}$  is the boundary for how far off our approximation is.

4. Let  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$ . Show that  $1 - \frac{1}{3!}$  approximates  $f(1)$  with an error less than 0.01.

$$|S - S_n| \leq |a_{n+1}|$$

$$\left| S - \left(1 - \frac{1}{3!}\right) \right| \leq \frac{1^4}{5!} \leq \frac{1}{120} \approx 0.0083\bar{3} \leq 0.01 \checkmark$$

$$\frac{1}{120} \leq \frac{1}{100} \checkmark$$

5. If the infinite series  $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4}{n}$  is approximated by  $P_k = \sum_{n=1}^k (-1)^{n+1} \frac{4}{n}$ , what is the least value of  $k$  for

which the alternating series error bound guarantees that  $|S - P_k| < \frac{7}{100}$ ? *error less than  $\frac{7}{100}$*

$$\frac{4}{k+1} < \frac{7}{100}$$

$$7(k+1) > 400$$

$$k+1 > \frac{400}{7}$$

$$k > \frac{400}{7} - 1$$

$K > 56.1428$

- (A) 55
- (B) 56
- (C) 57
- (D) 60

6. If the series  $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^3}$  is approximated by the partial sum  $S_k = \sum_{n=1}^k (-1)^{n+1} \frac{1}{n^3}$ , what is the least value

of  $k$  for which the alternating series error bound guarantees that  $|S - S_k| \leq \frac{7}{10000}$ ? *error less than  $\frac{7}{10000}$*

$$\frac{1}{(k+1)^3} < \frac{7}{10000}$$

$$7(k+1)^3 > 10000$$

$$(k+1)^3 > \frac{10000}{7}$$

$$k+1 > \sqrt[3]{\frac{10000}{7}}$$

$K > 10.262$

- (A) 10
- (B) 11
- (C) 12
- (D) 13

7. The series  $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$  converges by the alternating series test. If  $S_n = \sum_{k=1}^n (-1)^{k+1} a_k$  is the  $n$ th partial sum of the series, which of the following statements must be true?

- (A)  $\lim_{n \rightarrow \infty} S_n = 0$
- (B)  $\lim_{n \rightarrow \infty} a_n = S$
- (C)  $|S - S_{20}| \leq a_{26}$
- (D)  $|S - S_{25}| \leq a_{26}$



# Alternating Series Error Bound

## Practice

Calculus

A calculator may be used on all problems in this practice. For 1-2, approximate an interval of the sum of the alternating series using the Alternating Series Error Bound for the first 6 terms.

\*select Math → 0

$$1. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{3^n}$$

$$\sum_{x=1}^6 a_n \approx 0.185185$$

$$|a_7| = \frac{7}{3^7} \approx 0.0032$$

$$\text{Sum is } 0.185185 \pm 0.0032$$

$$\boxed{0.1819 \leq S \leq 0.188}$$

$$2. \sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{\ln(n+2)}$$

$$\sum_{x=1}^6 a_n \approx 1.14046$$

$$|a_7| = \frac{4}{\ln(9)} \approx 1.8204$$

$$S = 1.14046 \pm 1.8204$$

$$\boxed{-0.68 \leq S \leq 2.9609}$$

3. Determine the number of terms needed to approximate the sum of the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  with an error less than  $10^{-3}$ .

$$|a_{n+1}| < 10^{-3}$$

$$\frac{1}{(n+1)^2} < \frac{1}{1000}$$

$$(n+1)^2 > 1000$$

$$n+1 > \sqrt{1000}$$

$$n+1 > 31.6227$$

$$n > 30.6227$$

$$\boxed{n = 31 \text{ or more terms}}$$

4. The series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$  converges to  $S$ . Using the alternating series bound, what is the least number of terms that must be summed to guarantee a partial sum that is within 0.05 of  $S$ ?

$$|a_{n+1}| < 0.05$$

$$\frac{1}{\sqrt{n+1}} < 0.05$$

$$\sqrt{n+1} > \frac{1}{0.05}$$

$$\sqrt{n+1} > 20$$

$$n+1 > 400$$

$$n > 399$$

(A) 20

(B) 55

(C) 399

$\boxed{\text{(D) 400}}$

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8. If the series  $\sum_{n=1}^{\infty} (-1)^n \frac{1}{5n+1}$  is approximated by the partial sum with 15 terms, what is the alternating series error bound?

$$|a_{n+1}| = \frac{1}{5(n+1)+1} \rightarrow \frac{1}{5(16)+1} = \boxed{\frac{1}{81}}$$

- (A)  $\frac{1}{15}$                       (B)  $\frac{1}{16}$                       (C)  $\frac{1}{76}$                       **(D)  $\frac{1}{81}$**

9. The function  $f$  is defined by the power series  $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x}{(2n+1)!}$  for all real numbers  $x$ . Show that  $1 - \frac{1}{3!} + \frac{1}{5!}$  approximates  $f(1)$  with an error less than  $\frac{1}{4000}$ .

$$|S - S_n| < |a_{n+1}|$$

$$\left| f(1) - \left[ 1 - \frac{1}{3!} + \frac{1}{5!} \right] \right| < \frac{1}{7!}$$

$\frac{1}{7!} = \frac{1}{5040}$  which is less than  $\frac{1}{4000}$

↖ next term in the series  
(error bound for partial sum  $(1 - \frac{1}{3!} + \frac{1}{5!})$ )

**Alternating Series Error Bound**

**Test Prep**

10. **Calculator active!** Let  $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$  for all  $x$  for which the series converges.

a. Use the first three terms of the series to approximate  $f(-\frac{1}{3})$ .

$$f(-\frac{1}{3}) \approx \sum_{n=1}^3 \frac{(-\frac{1}{3})^n n^n}{n!} = -0.2778$$

b. How far off is this estimate from the value of  $f(-\frac{1}{3})$ ? Justify your answer.

$$a_4 = \frac{(-\frac{1}{3})^4 \cdot 4^4}{4!} = 0.13168$$

The estimate of  $-0.2778$  is off by at most  $0.13168$  or less.

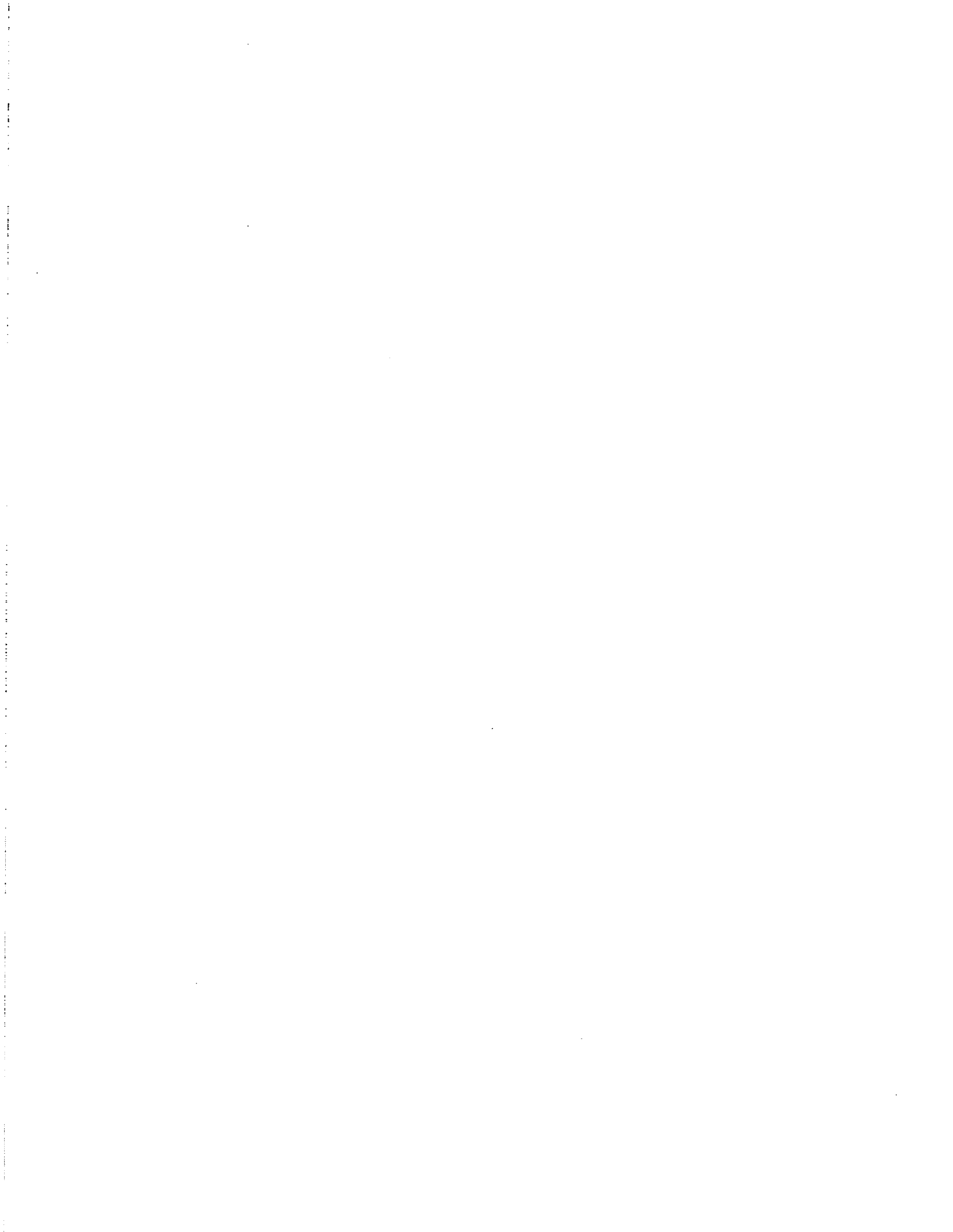
$a_4$  is the error bound for the partial sum  $S_3$

(the next unused term)

11. If the series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$  is approximated with the series  $\sum_{n=1}^7 (-1)^{n+1} \frac{1}{n^2}$ , what is the error bound?

Error Bound for partial sum  $S_7$  is  $|a_8| \leftarrow 8^{th} \text{ term}$

$$|a_8| = \frac{1}{8^2} = \boxed{\frac{1}{64}}$$



# BC Calculus – 10.10a Notes - Finding Taylor Polynomial Approximations of Functions

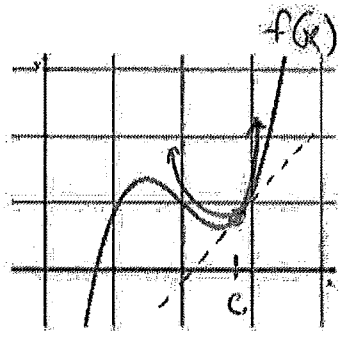
Taylor polynomials are created to help us approximate other functions. Why would we do this? Because polynomials are easy to work with in calculus (i.e., taking a derivative or integral).

A Maclaurin polynomial is a special type of Taylor polynomial.

To start, we choose an  $x$ -value to center our polynomial approximation. Let's call that  $x = c$ . Our approximation will have the same  $y$ -value as the original function at  $x = c$ .

$$f(c) = p(c)$$

We expand the approximation about  $x = c$ . Another way of saying this: "the functions are centered at  $x = c$ ."



Explore with an example:  $f(x) = e^x$ . Let  $c = 0$ .

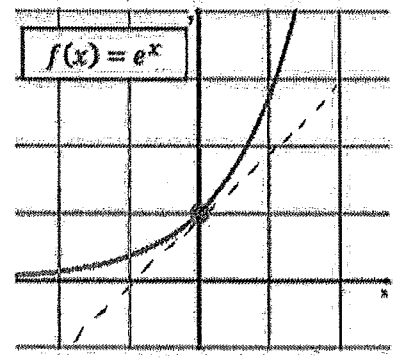
We know  $f(0) = 1$ . We need  $p(0) = 1$

We want to make the graphs have a similar shape at the point  $x = c$ . They should have the same slope. That leads us to

$$f'(c) = p'(c)$$

In this example, that means  $f'(0) = p'(0)$

The polynomial approximation will look like this:



$$y - y_1 = m(x - x_1)$$

$$p(x) - f(c) = f'(c)(x - c) \leftarrow \text{point-slope form}$$

Or rewritten:

$$* f'(x) = e^x$$

$$f'(0) = e^0 = 1$$

$$p(x) = f(c) + f'(c)(x - c)$$

$$p(x) = f(0) + f'(0)(x - 0)$$

$$p(x) = 1 + 1x$$

$$p(x) = 1 + x$$

This is called a first order approximation. It works for a small interval around our point of center.

To improve the approximation, make the second derivatives agree at  $x = c$ .

We want  $f(c) = p(c)$ ,  $f'(c) = p'(c)$ ,  $f''(c) = p''(c)$ .

For our example this is  $f(0) = p(0)$ ,  $f'(0) = p'(0)$ ,  $f''(0) = p''(0)$

If we worked through a similar process, we'd end up with the following:

Second-order approximation: 
$$p(x) = 1 + x + \frac{1}{2}x^2$$

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If we are centered at  $x = 0$ , then we have the following pattern:

$$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

**N<sup>th</sup> Taylor Polynomial**

If  $f(x)$  is a differentiable function, then an approximation of  $f$  centered about  $x = c$  can be modeled by

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots + \dots$$

where  $n$  is the order of the approximation.

$$\frac{f^{(n)}(c)(x-c)^n}{n!}$$

**Maclaurin Polynomial**

A Maclaurin polynomial is a Taylor polynomial centered about  $x = 0$ . It can be modeled by

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)x^n}{n!}$$

where  $n$  is the order of the approximation.

1. Find the third-degree Maclaurin polynomial for  $f(x) = e^{2x}$

$$P_3(x) = f(0) + f'(0) \cdot x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$P_3(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}$$

$f(x) = e^{2x}$	$f(0) = 1$
$f'(x) = 2e^{2x}$	$f'(0) = 2$
$f''(x) = 2e^{2x} \cdot 2$	$f''(0) = 4$
$f'''(x) = 8e^{2x}$	$f'''(0) = 8$

$$P_3(x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3$$

Evaluate at  $f(0.2)$  and  $p_3(0.2)$

$$f(0.2) = e^{2(0.2)} \approx 1.4918$$

$$P_3(0.2) \approx 1.4906$$

this is a Maclaurin polynomial approximation of the actual y-value for  $f(0.2)$

2. Find a fourth-degree Taylor Polynomial for  $f(x) = \ln x$  centered at  $x = 1$ .

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^{(4)}(1)(x-1)^4}{4!}$$

$$P_4(x) = 0 + 1(x-1) + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-6(x-1)^4}{4!}$$

$$P_4(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

Evaluate at  $f(1.1)$  and  $p_4(1.1)$

$$f(1.1) = 0.0953101798$$

$$P_4(1.1) = 0.0953083333$$

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x} = x^{-1}$	$f'(1) = 1$
$f''(x) = -1x^{-2}$	$f''(1) = -1$
$f'''(x) = +2x^{-3}$	$f'''(1) = 2$
$f^{(4)}(x) = -6x^{-4}$	$f^{(4)}(1) = -6$



**Coefficients of a Taylor Polynomial**

The coefficient of the  $n$ th degree term in a Taylor polynomial for a function  $f$  centered at  $x = c$  is

$$\frac{f^n(c)}{n!}$$

3. Let  $f$  be a function with third derivative  $f'''(x) = (8x + 2)^{3/2}$ . What is the coefficient of  $(x - 2)^4$  in the fourth-degree Taylor Polynomial for  $f$  about  $x = 2$ .

$$f^4(x) = \frac{3}{2}(8x+2)^{1/2}(8) \quad \left| \quad f^4(2) = 12 \cdot \sqrt{9} \cdot 2 \right.$$

$$f^4(2) = \frac{3}{2}(18)^{1/2} \cdot 8 \quad \left| \quad f^4(2) = 36\sqrt{2} \right.$$

$$\frac{36\sqrt{2}}{4!} \rightarrow \frac{36\sqrt{2}}{4 \cdot 3 \cdot 2 \cdot 1}$$

$\frac{3\sqrt{2}}{2}$

Practice Problems

$c = 0$

1. Find the fourth-degree Maclaurin Polynomial for  $e^{4x}$ .

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$f(x) = e^{4x} \quad f(0) = e^0 = 1$$

$$f'(x) = 4e^{4x} \quad f'(0) = 4$$

$$f''(x) = 4^2 e^{4x} \quad f''(0) = 4^2$$

$$f'''(x) = 4^3 e^{4x} \quad f'''(0) = 4^3$$

$$f^{(4)}(x) = 4^4 e^{4x} \quad f^{(4)}(0) = 4^4$$

$$P_4(x) = 1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \frac{4^4}{4!}x^4$$

$P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4$

$c = 0$

2. Find the fifth-degree Maclaurin Polynomial for the function  $f(x) = \sin x$ .

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$f = \sin(x) \quad f(0) = 0$$

$$f' = \cos(x) \quad f'(0) = 1$$

$$f'' = -\sin(x) \quad f''(0) = 0$$

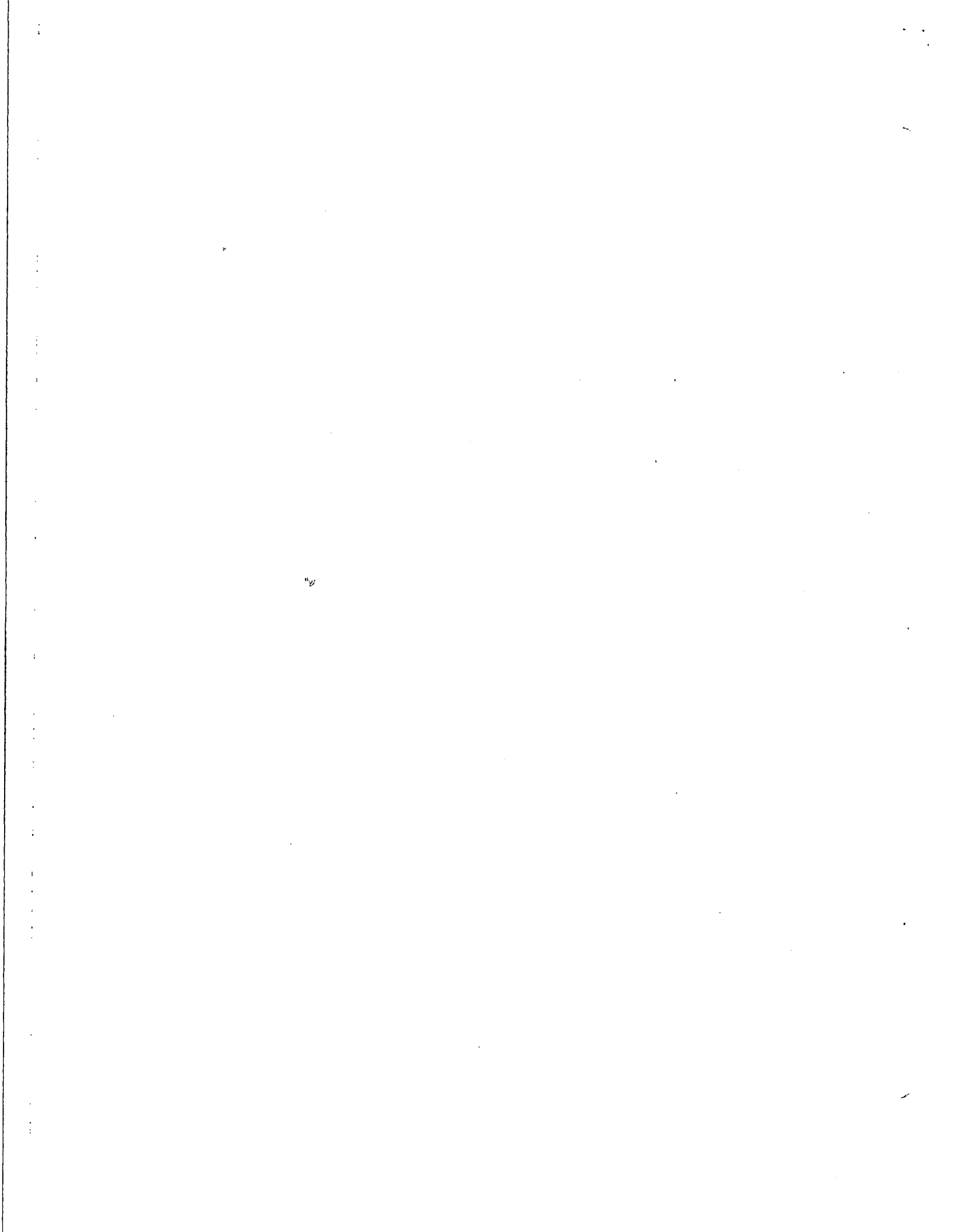
$$f''' = -\cos(x) \quad f'''(0) = -1$$

$$f^{(4)} = \sin(x) \quad f^{(4)}(0) = 0$$

$$f^{(5)} = \cos(x) \quad f^{(5)}(0) = 1$$

$$0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$



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\* P\_n(x) = f^n(c) / n! (x-c)^n

3. Find the third-degree Taylor Polynomial for f(x) = ln(2x) about x = 1.

c = 1

P\_3(x) = f(1) + f'(1)(x-1) + f''(1)/2!(x-1)^2 + f'''(1)/3!(x-1)^3

f = ln(2x) f(1) = ln 2
f' = 2/(2x) = 1/x = x^-1 f'(1) = 1
f'' = -1x^-2 f''(1) = -1
f''' = 2x^-3 f'''(1) = 2

P\_3(x) = ln 2 + 1(x-1) + -1/2(x-1)^2 + 2/3!(x-1)^3

P\_3(x) = ln 2 + (x-1) - 1/2(x-1)^2 + 1/3(x-1)^3

4. Find the third-degree Taylor Polynomial about x = 0 for ln(1-x).

c = 0

P\_3(x) = f(0) + f'(0)(x-0) + f''(0)/2!(x-0)^2 + f'''(0)/3!(x-0)^3

f = ln(1-x) f(0) = ln 1 = 0
f' = -1/(1-x) = -(1-x)^-1 f'(0) = -1
f'' = 1(1-x)^-2 (-1) f''(0) = -1
f''' = 2(1-x)^-3 (-1) f'''(0) = -2

P\_3(x) = 0 + -1(x-0) - 1/2!(x-0)^2 - 2/3!(x-0)^3

P\_3(x) = -1(x-0) - 1/2(x-0)^2 - 1/3(x-0)^3

P\_3(x) = -1x - 1/2x^2 - 1/3x^3

5. Find the third-degree Taylor Polynomial about x = 4 of sqrt(x).

c = 4

P\_3(x) = f(4) + f'(4)(x-4) + f''(4)/2!(x-4)^2 + f'''(4)/3!(x-4)^3

f = x^1/2 f(4) = sqrt(4) = 2
f' = 1/2 x^-1/2 f'(4) = 1/(2\*sqrt(4)) = 1/4
f'' = -1/4 x^-3/2 f''(4) = -1/(4\*(4)^(3/2)) = -1/(4\*8) = -1/32
f''' = 3/8 x^-5/2 f'''(4) = 3/(8\*(4)^(5/2)) = 3/(8\*32) = 3/256

P\_3(x) = 2 + 1/4(x-4) - 1/32 \* 1/2 (x-4)^2 + 3/256 \* 1/3! (x-4)^3

P\_3(x) = 2 + 1/4(x-4) - 1/64 (x-4)^2 + 1/512 (x-4)^3

6. The function f has derivatives of all orders for all real numbers with f(1) = -1, f'(1) = 4, f''(1) = 6, and f'''(1) = 12. Using the third-degree polynomial for f about x = 1, what is the approximation of f(1.1)?

c = 1

f(x) = f(1) + f'(1)(x-1) + f''(1)/2!(x-1)^2 + f'''(1)/3!(x-1)^3

f(x) = -1 + 4(x-1) + 6/2!(x-1)^2 + 12/3!(x-1)^3

f(x) = -1 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3

f(1.1) = -1 + 4(0.1) + 3(0.1)^2 + 2(0.1)^3

f(1.1) = -0.568

f(0)=3, f'(0)=4

7. A function f has a Maclaurin series given by 3 + 4x + 2x^2 + 1/3 x^3 + ... and the series converges to f(x) for all real numbers x. If g is the function defined by g(x) = e^{f(x)}, what is the coefficient of x in the Maclaurin series for g? (c=0)

P\_n(x) = g(0) + g'(0)(x-0) + g''(0)/2(x-0)^2

g'(x) = e^{f(x)} \* f'(x)

g'(0) = e^{f(0)} \* f'(0) -> e^3 \* 4 -> 4e^3

8. Let f be a function with third derivative f'''(x) = (7x + 2)^{3/2}. What is the coefficient of (x - 2)^4 in the fourth-degree Taylor Polynomial for f about x = 2?

f''(x) = 3/2 (7x+2)^{1/2} (7)

f''(2) = 3/2 (16)^{1/2} (7)

= 3/2 (4)(7) = 42

f''(2) / 4! (x-2)^4

42 / 24 = 7/4

9. Let P(x) = 4x^2 - 6x^3 + 8x^4 + 4x^5 be the fifth-degree Taylor Polynomial for the function f about x = 0. What is the value of f'''(0)? (c=0)

f'''(0) / 3! (x-0)^3

f'''(0) / 3! = -6

f'''(0) = -6 \* 3!

f'''(0) = -36

10. Let P be the second-degree Taylor Polynomial for f(x) = e^{-3x} about x = 3. What is the slope of the line tangent to the graph of P at x = 3?

at x=3, P'(3) = f'(3)

f'(3) = -3e^{-9} = -3/e^9

f'(x) = e^{-3x} (-3)

f'(3) = e^{-3(3)} \* -3

11. Let f be a function with f(4) = 2, f'(4) = -1, f''(4) = 6, and f'''(4) = 12. What is the third-degree Taylor Polynomial for f about x = 4? (c=4)

P\_3(x) = f(4) + f'(4)(x-4) + f''(4)/2!(x-4)^2 + f'''(4)/3!(x-4)^3

P\_3(x) = 2 - 1(x-4) + 6/2(x-4)^2 + 12/3!(x-4)^3

P\_3(x) = 2 - 1(x-4) + 3(x-4)^2 + 2(x-4)^3

12. Let  $f$  be a function that has derivatives of all orders for all real numbers. Assume  $f(1) = 3, f'(1) = -2, f''(1) = 2,$  and  $f'''(1) = 4.$  Use a second-degree Taylor Polynomial to approximate  $f(0.7).$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$P_2(x) = 3 - 2(x-1) + \frac{2}{2!}(x-1)^2$$

$$f(0.7) \approx P_2(0.7) = 3 - 2(-0.3) + (-0.3)^2 = 3.69$$

13. The function  $f$  has derivatives of all orders for all real numbers with  $f(0) = 4, f'(0) = -3, f''(0) = 3,$  and  $f'''(0) = 2.$  Let  $g$  be the function given by  $g(x) = \int_0^x f(t) dt.$  Find the third-degree Taylor Polynomial for  $g$  about  $x = 0, c = 0$

$$P_3(x) = g(0) + g'(0)(x-0) + \frac{g''(0)}{2!}(x-0)^2 + \frac{g'''(0)}{3!}(x-0)^3$$

$$g(x) = \int_0^x f(t) dt \quad g(0) = 0$$

$$g'(x) = f(x) \quad g'(0) = f(0) = 4$$

$$g''(x) = f'(x) \quad g''(0) = f'(0) = -3$$

$$g'''(x) = f''(x) \quad g'''(0) = f''(0) = 3$$

$$P_3(x) = 0 + 4x - \frac{3}{2}x^2 + \frac{3}{3!}x^3$$

$$P_3(x) = 4x - \frac{3}{2}x^2 + \frac{1}{2}x^3$$

14.

$x$	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
-4	1	-2	-4	2

Selected values for  $f(x)$  and its first three derivatives are shown in the table above. What is the approximation for the value of  $f(-3)$  about  $x = -4$  obtained using the third-degree Taylor Polynomial for  $f.$

$$P_3(x) = f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3$$

$$P_3(x) = 1 - 2(x+4) - \frac{4}{2}(x+4)^2 + \frac{2}{3!}(x+4)^3$$

$$f(-3) \approx P_3(-3) = 1 - 2(1) - 2(1)^2 + \frac{2}{6}(1)^3 = \frac{-8}{3} \approx -2.667$$

**Taylor Polynomial Approximations**

**Test Prep**

15. Which of the following polynomial approximations is the best for  $\cos(3x)$  near  $x = 0$ ?

$$f(x) = \cos(3x) \quad f(0) = \cos 0 = 1$$

$$f'(x) = -\sin(3x) \cdot 3 \quad f'(0) = 0$$

$$f''(x) = -3\cos(3x) \cdot 3 \quad f''(0) = -9$$

$$P(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$$

$$P(x) = 1 + 0(x) - \frac{9}{2}x^2$$

(A)  $1 + \frac{3}{2}x$

(B)  $1 - \frac{9}{2}x^2$

(C)  $1 + x$

(D)  $1 - \frac{9}{2}x + x^2$

13

16. Consider the logistic differential equation  $\frac{dy}{dt} = \frac{y}{6}(4-y)$ . Let  $y = f(t)$  be the particular solution to the differential equation with  $f(0) = 6$ .

a. Write the second-degree Taylor Polynomial for  $f$  about  $t = 0$ .

$$P_2(t) = f(0) + f'(0)(t-0) + \frac{f''(0)}{2!}(t-0)^2$$

$$f(0) = 6$$

$$f'(0) = \frac{6}{6}(4-6) = -2$$

$$f''(x) = \frac{1}{6} \left( \frac{dy}{dt} \right) (4-y) + \frac{y}{6} (-1) \left( \frac{dy}{dt} \right)$$

$$f''(0) = \frac{1}{6}(-2)(4-6) + \frac{6}{6}(-1)(-2)$$

$$= \frac{2}{3} + 2 = \frac{8}{3}$$

$$P_2(t) = 6 - 2(t-0) + \frac{8}{3} \cdot \frac{1}{2} (t)^2$$

$$P_2(t) = 6 - 2t + \frac{4}{3}t^2$$

b. Use the results from part a to approximate  $f(1)$ .

$$f(1) \approx 6 - 2(1) + \frac{4}{3}(1) = \boxed{\frac{16}{3}}$$

t (seconds)	0	4	10
$x'(t)$ meters per second	5.0	5.8	4.0

17. The position of a particle moving along a straight line is modeled by  $x(t)$ . Selected values of  $x'(t)$  are shown in the table above and the position of the particle at time  $t = 10$  is  $x(10) = 8$ .

a. Approximate  $x''(8)$  using the average rate of change of  $x'(t)$  over the interval  $4 \leq t \leq 10$ . Show computations that lead to your answer.

$$\frac{x'(10) - x'(4)}{10 - 4} = \frac{4.0 - 5.8}{10 - 4} = \frac{-1.8}{6} = \boxed{-0.3}$$

b. Using correct units, explain the meaning of  $x''(8)$  in the context of the problem.

$x''(8)$  is the rate at which velocity is changing (which is acceleration) at  $t = 8$  with units of meters/sec<sup>2</sup>

c. Use a right Riemann sum with two subintervals to approximate  $\int_0^{10} |x'(t)| dt$ .

$$\int_0^{10} |x'(t)| dt \approx 4(5.8) + 6(4) = \boxed{47.2}$$

↖ Absolute value

d. Let  $s$  be a function such that the third derivative of  $s$  with respect to  $t$  is  $(t-3)^7$ . Write the fourth-degree term of the fourth-degree Taylor Polynomial for  $s$  about  $t = 1$ .

$$f'''(x) = 7(t-3)^6$$

$$f'''(1) = 7(1-3)^6 = 7(-2)^6 = 448$$

4<sup>th</sup> term is

$$\frac{f'''(1)}{4!} (t-1)^4 \rightarrow \frac{448}{4!} (t-1)^4$$

$$\boxed{\frac{56}{3} (t-1)^4}$$

BC Calculus – 10.8a Notes – Radius and Interval of Convergence of Power Series

Power Series

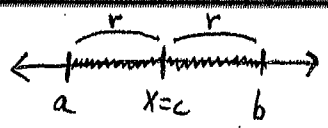
$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n \quad \text{centered at } x=0$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + a_3 (x-c)^3 + \dots + a_n (x-c)^n$$

The domain of a power series is the set of all  $x$ -values for which the power series converges.

Note! The center is always part of the domain.

**Three ways a power series may converge:**



1. Converges to an interval
  - a. The radius is the distance from the center to the edge of interval
2. Converges to all real numbers
3. Converges to the center ( $x=c$ ) only

The Interval of Convergence is the set of values for convergence. We use the Ratio Test to find the interval of convergence.

**Ratio Test for Interval of Convergence**

If you have a power series  $\sum_{n=1}^{\infty} a_n$ , find  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ .

- $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then the series converges on an interval
- $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ , then the series converges for all values of  $x$
- $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then the series converges only to center ( $x=c$ ) (diverges everywhere else!)

Find the radius and interval of convergence.

1.  $\sum_{n=1}^{\infty} \frac{n}{3^n} (x-5)^n$   $\lim_{n \rightarrow \infty} \left| \frac{n+1}{3^{n+1}} \cdot \frac{(x-5)^{n+1}}{n(x-5)^n} \cdot 3^n \right|$

$\lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \cdot \frac{(x-5)}{3} \right| < 1 \rightarrow \left| \frac{x-5}{3} \right| < 1$

$-1 < \frac{x-5}{3} < 1$   $|x-c| < r$

$-3 < x-5 < 3$   
 $2 < x < 8$

test  $x=8$   $|x-5| < 3$   
 $\sum_{n=1}^{\infty} \frac{n}{3^n} (3)^n$  diverges  
 (nth term test)

\* check convergence at endpoints!!

at  $x=2$   $\sum_{n=1}^{\infty} \frac{n}{3^n} (-3)^n = (-1)^n n$  diverges

Interval of convergence  
 IOC:  $2 < x < 8$   
 Radius:  $r=3$

2.  $\sum_{n=0}^{\infty} 3(x-2)^n$   $\lim_{n \rightarrow \infty} \left| 3(x-2)^{n+1} \cdot \frac{1}{3(x-2)^n} \right|$

$\lim_{n \rightarrow \infty} |x-2| < 1$   $-1 < x-2 < 1$

$|x-c| < r$   $1 < x < 3$

\* check endpoints:

at $x=1$ $\sum_{n=0}^{\infty} 3(1-2)^n$ $\sum_{n=0}^{\infty} 3(-1)^n$ diverges	at $x=3$ $\sum_{n=0}^{\infty} 3(3-2)^n$ $\sum_{n=0}^{\infty} 3(1)^n$ diverges
---	--

IOC:  $1 < x < 3$   
 Radius:  $r=1$

3.  $\sum_{n=1}^{\infty} \frac{(x+2)^{n+1}}{n^3}$   $\lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+2}}{(n+1)^3} \cdot \frac{n^3}{(x+2)^{n+1}} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{(x+2) \cdot n^3}{(n+1)^3} \right| < 1$

$|x+2| < 1$   
 $-1 < x+2 < 1$   
 $-3 < x < -1$

test  $x=-3$   
 $\sum \frac{(-1)^{n+1}}{n^3}$   
 converges

test  $x=-1$   
 $\sum \frac{(1)^{n+1}}{n^3}$   
 converges

IOC:  $-3 \leq x \leq -1$   
 Radius:  $r=1$

4.  $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$   $\lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)! x^{2n}} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(2n)! \cdot x^{2n+2} \cdot n!}{(n+1)n \cdot (2n)! \cdot x^{2n}} \right| = \infty$

Converges only at  $x=0$   
 Radius = 0

since there are no range of values

5.  $\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$

$\lim_{n \rightarrow \infty} \left| \frac{x^{3n+3}}{(n+1)!} \cdot \frac{n!}{x^{3n}} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{x^3 \cdot n!}{(n+1) \cdot n!} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{1}{n!} \cdot x^3 \right| = 0$

converges for all values of  $x$   
 IOC:  $(-\infty, \infty)$   
 radius =  $\infty$



Radius and Interval of Convergence of Power Series

Practice

Calculus

Find the interval of convergence for each power series.

1.  $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$   $\lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(x-1)^n} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{x-1}{4} \right| < 1 \rightarrow -1 < \frac{x-1}{4} < 1$

$-4 < x-1 < 4$

$-3 < x < 5$

\*check endpoints to see if they converge or diverge.

test  $x = -3$

$\sum \frac{(-4)^n}{4^n} \rightarrow \sum (-1)^n$  diverges

$-3 < x < 5$

test  $x = 5$

$\sum \frac{4^n}{4^n} \rightarrow \sum (1)^n$  diverges

2.  $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$   $\lim_{n \rightarrow \infty} \left| \frac{(x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x+2)^n} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{x+2}{3} \right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1$

$-3 < x+2 < 3$

$-5 < x < 1$

test  $x = -5$

$\sum \frac{(-3)^n}{3^n} \rightarrow \sum (-1)^n$  diverges

test  $x = 1$

$\sum \frac{(3)^n}{3^n} \rightarrow \sum (1)^n$  diverges

$-5 < x < 1$

3.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-2)^n}{n 2^n}$   $\lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1) 2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-2)^n} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{(x-2)}{2} \cdot \frac{n}{n+1} \right| < 1 \rightarrow -1 < \frac{x-2}{2} < 1$

$-2 < x-2 < 2$

$0 < x < 4$

test  $x = 0$

$\sum \frac{(-1)^{n+1} (-2)^n}{n (2^n)}$

$\sum \frac{(-1)^{n+1} (-1)^n}{n}$  diverges

test  $x = 4$

$\sum \frac{(-1)^{n+1} (2)^n}{n \cdot 2^n}$  converges by AST

$0 < x < 4$

4.  $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$   $\lim_{n \rightarrow \infty} \left| \frac{[2(n+1)]! \left(\frac{x}{3}\right)^{n+1}}{(2n)! \left(\frac{x}{3}\right)^n} \right|$

$\lim_{n \rightarrow \infty} \left| \frac{(2n+2)! \left(\frac{x}{3}\right)}{(2n)!} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \left(\frac{x}{3}\right) \right|$

$= \infty$

Converges only at center  $x = 0$

Find the radius of convergence for each series.

5.  $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$   $\lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x)^n} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{(4x)^{n+1} \cdot (4x)}{(4x)^n \cdot (n+1)^2} \cdot \frac{n^2}{(4x)^n} \right| = \lim_{n \rightarrow \infty} |4x| < 1$

$|4x| < 1 \rightarrow -1 < 4x < 1$   
 $-\frac{1}{4} < x < \frac{1}{4} \rightarrow |x| < \frac{1}{4}$

Radius = 1/4

6.  $\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{2 \cdot 3^{n+1}}$   $\lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+2}}{2 \cdot 3^{n+2}} \cdot \frac{2 \cdot 3^{n+1}}{(x-4)^{n+1}} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{(x-4)}{3} \right| < 1$

$|x-4| < 3$   
 $* \uparrow$   
 $|x-c| < r$

radius = 3

7.  $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$   $\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}}{[2(n+1)]!} \cdot \frac{(2n)!}{x^{2n}} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{x^2 \cdot (2n)!}{(2n+2)!} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^2 \cdot (2n)!}{(2n+2)(2n+1)(2n)!} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0$  (always less than 1)

Radius =  $\infty$

8.  $\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$   $\lim_{n \rightarrow \infty} \left| \frac{(2(n+1))! x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{(2n)! x^{2n}} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{(2n+2)! x^{2n+2} \cdot n!}{(n+1)! (2n)! x^{2n}} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{(2n+2)(2n+1)(2n)! x^{2n+2} \cdot n!}{(n+1)n! (2n)! x^{2n}} \right| = \infty$

Radius = 0

\* converges only at center (c=0)

What are all values of x for which each series converges?

9.  $\sum_{n=1}^{\infty} \left( \frac{4}{x^2+1} \right)^n$   $x > \sqrt{3}$  or  $x < -\sqrt{3}$   
 $\lim_{n \rightarrow \infty} \left| \left( \frac{4}{x^2+1} \right)^{n+1} \cdot \left( \frac{x^2+1}{4} \right)^n \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{4}{x^2+1} \right| < 1$   
 $4 < (x^2+1)$   
 $x^2+1 > 4$   
 $x^2 > 3$   
 \* test endpoints:  
 at  $x = \sqrt{3}$ :  
 $\sum \left( \frac{4}{\sqrt{3}^2+1} \right)^n$  diverges  
 at  $x = -\sqrt{3}$ :  
 $\sum \left( \frac{4}{\sqrt{3}^2+1} \right)^n$  diverges  
 converges when  $x > \sqrt{3}$  or  $x < -\sqrt{3}$

10.  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( x + \frac{3}{2} \right)^n$   $\lim_{n \rightarrow \infty} \left| \frac{(x+\frac{3}{2})^{n+1}}{n+1} \cdot \frac{n}{(x+\frac{3}{2})^n} \right|$   
 $\lim_{n \rightarrow \infty} \left| \frac{(x+\frac{3}{2})}{1} \right| < 1$   $|x+\frac{3}{2}| < 1$   
 $-1 < x+\frac{3}{2} < 1$  \*  $|x-c| < r$   $r=1$   
 $-\frac{5}{2} < x < -\frac{1}{2}$   
 test  $x = -\frac{5}{2}$ :  
 $\sum \frac{(-1)^n}{n} \left( -\frac{5}{2} + \frac{3}{2} \right)^n$   
 $\sum \frac{(-1)^n}{n} (-1)^n$   
 diverges (p-series)  
 test  $x = -\frac{1}{2}$ :  
 $\sum \frac{(-1)^n}{n} \cdot (1)^n$   
 converges by AST  
 $-\frac{5}{2} < x \leq -\frac{1}{2}$

$$11. \sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n} \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-2)}{3} \cdot \frac{n}{n+1} \right| < 1 \quad \left| \frac{x-2}{3} \right| < 1$$

$$|x-2| < 3$$

$$-3 < x-2 < 3$$

$$-1 < x < 5$$

test  $x=5$ :

$$\sum \frac{(5-2)^n}{n \cdot 3^n} \rightarrow \sum \frac{(3)^n}{n \cdot 3^n}$$

diverges (p-series)

test  $x=-1$ :

$$\sum \frac{(-1-2)^n}{n \cdot 3^n}$$

$$\sum \frac{(-3)^n}{n \cdot 3^n} \rightarrow \sum \frac{(-1)^n}{n} \text{ converges by AST}$$

$$\boxed{-1 \leq x < 5}$$

$$12. \sum_{n=0}^{\infty} \frac{x^{5n}}{n!} \lim_{n \rightarrow \infty} \left| \frac{x^{5(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x^5 \cdot n!}{(n+1) \cdot n!} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^5}{n+1} \right| = 0 \text{ (always less than 1)}$$

converges for all values of  $x$ :

$$\boxed{(-\infty, \infty)}$$

**16.13 Radius and Interval of Convergence of Power Series**

**Test Prep**

13. The radius of convergence for the power series  $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$  is equal to 1. What is the interval of convergence?

$$|x-c| < r$$

$$c=4, r=1$$

$$|x-4| < 1$$

$$-1 < x-4 < 1$$

$$3 < x < 5$$

\*check endpoints:

test  $x=3$ :

$$\sum \frac{(3-4)^{2n}}{n} \rightarrow \sum \frac{(-1)^{2n}}{n}$$

diverges by p-series

test  $x=5$ :

$$\sum \frac{(5-4)^{2n}}{n}$$

$$\sum \frac{(1)^{2n}}{n}$$

diverges by p-series

$$\boxed{3 < x < 5}$$

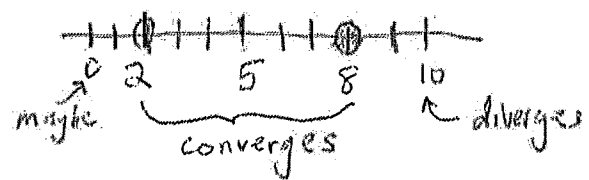
14. If the power series  $\sum_{n=0}^{\infty} a_n(x-5)^n$  converges at  $x=8$  and diverges at  $x=10$ , which of the following must be true?

\*center at  $x=5$

maybe I. The series converges at  $x=2$ . \*Radius is between 3 and 5

yes II. The series converges at  $x=3$ .

maybe III. The series diverges at  $x=0$ .



(A) I only

**(B) II only**

(C) I and II only

(D) II and III only

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15. The coefficients of the power series  $\sum_{n=0}^{\infty} a_n(x-3)^n$  satisfy  $a_0 = 6$  and  $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$  for all  $n \geq 1$ . What is the radius of convergence?

$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = ?$ 
 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}(x-3)^{n+1}}{a_n(x-3)^n} \right|$ 
 $\lim_{n \rightarrow \infty} \left| \frac{2}{3} \cdot (x-3) \right| < 1$

$b_n = a_n(x-3)^n$   
 $b_{n+1} = a_{n+1}(x-3)^{n+1}$

$a_{n+1} = \left(\frac{2(n+1)+1}{3(n+1)+1}\right) \cdot a_n$   
 $a_{n+1} = \frac{2n+3}{3n+4} a_n$

$\frac{a_{n+1}}{a_n} = \frac{2n+3}{3n+4}$   
 $\lim_{n \rightarrow \infty} \left| \frac{2n+3}{3n+4} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \right|$

$-1 < \frac{2}{3}(x-3) < 1$   
 $-\frac{3}{2} < x-3 < \frac{3}{2}$   
 $\frac{3}{2} < x < \frac{9}{2}$

$\left| \frac{2}{3}(x-3) \right| < 1$   
 $|x-3| < \frac{3}{2}$   
 $|x-c| < r$

**Radius =  $\frac{3}{2}$**

16. The radius of convergence for the power series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n \cdot 5^n}$  is 5, what is the interval of convergence?

Center is  $x=5$   
 radius = 5  
 $|x-5| < 5$

$-5 < x-5 < 5$   
 $0 < x < 10$

test endpoints:  
 $x=0$ :  
 $\sum \frac{(-1)^{n+1}(-5)^n}{n \cdot 5^n} \rightarrow \frac{(-1)^n(-5)^n}{5^n} = \frac{(-1)^n(-5)^n}{5^n} = \frac{(-1)^n}{n}$  diverges

$x=10$ :  
 $\sum \frac{(-1)^{n+1}(5)^n}{n \cdot 5^n}$  converges by AST

**$0 < x \leq 10$**

(A)  $-5 < x < 5$       (B)  $-5 < x \leq 5$       (C)  $0 < x < 10$       **(D)  $0 < x \leq 10$**

17. Let  $a_n = \frac{1}{n \ln n}$  for  $n \geq 3$  and let  $f$  be the function given by  $f(x) = \frac{1}{x \ln x}$ .

a. The function  $f$  is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series  $\sum_{n=3}^{\infty} a_n$ .

$\int_3^{\infty} \frac{1}{x \ln x} dx$

$u = \ln x$   
 $\frac{du}{dx} = \frac{1}{x}$   
 $dx = x du$

$\lim_{b \rightarrow \infty} \int_3^b \frac{1}{x \cdot u} \cdot x du$   
 $\ln|u| \rightarrow \ln|\ln x|$   
 $\ln|\ln x| \Big|_3^b \rightarrow \lim_{b \rightarrow \infty} \ln|\ln b| - \ln|\ln 3|$   
 $\infty - \ln|\ln 3| = \infty$

**Series Diverges by Integral Test**

b. Find the interval of convergence of the power series  $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$

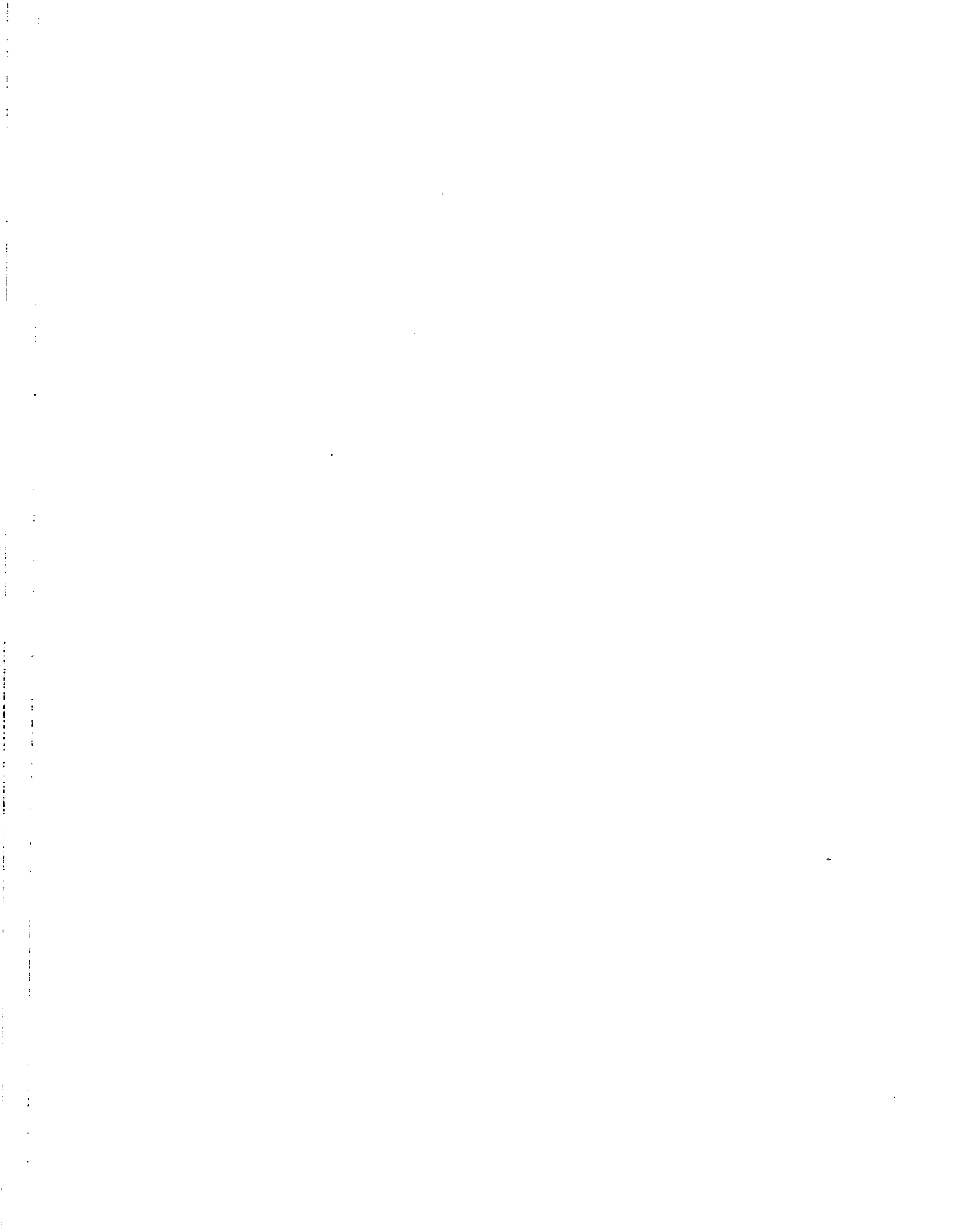
$$\lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+2}}{(n+1) \cdot \ln(n+1)} \cdot \frac{n \ln(n)}{(x-2)^{n+1}} \right| \quad \left| \lim_{n \rightarrow \infty} |x-2| < 1 \right.$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-2)}{1} \cdot \frac{n \ln(n)}{(n+1) \ln(n+1)} \right| < 1 \quad \left. \begin{array}{l} -1 < x-2 < 1 \\ 1 < x < 3 \end{array} \right.$$

\*test endpoints:

<u>x=1:</u>		<u>x=3</u>	
$\sum \frac{(-1)^{n+1}}{n \cdot \ln(n)}$	converges by AST	$\sum \frac{(1)^{n+1}}{(n) \ln(n)}$	diverges by Integral Test

$1 \leq x < 3$



BC Calculus – 10.9 Notes – Finding Taylor & Maclaurin Series for a Function

**Taylor Series**

If  $f(x)$  has derivatives of all orders at  $x = c$ , then a Taylor Series may be formed that is equal to the function for many common functions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

If  $c = 0$  it is a **Maclaurin Series**.

You need to know the following series:  $e^x$ ,  $\cos x$ ,  $\sin x$ ,  $\frac{1}{1+x}$

The Taylor series of these functions are exact when we go to  $\infty$ . They must be memorized!

Maclaurin Series for $e^x$	Maclaurin Series for $\sin x$
$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$ $e^x$ $e^0 = 1$ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad   \text{Ioc: } (-\infty, \infty)$	$f(x) = \sin x \quad f(0) = 0$ $f'(x) = \cos x \quad f'(0) = 1$ $f''(x) = -\sin x \quad f''(0) = 0$ $f'''(x) = -\cos x \quad f'''(0) = -1$ $f^{(4)}(x) = \sin x \quad f^{(4)}(0) = 0$ $\sin x = 0 + x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} - \frac{x^5}{5!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Memorize the following!

Function	Series (expanded)	Series Notation	Int. of Conv.
$e^x =$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin x =$ (odd)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x =$ (even)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1+x} =$	$1 - x + x^2 - x^3 + \dots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$

The function  $f(x) = \frac{1}{1-x}$  is actually a geometric series.

Recall:  $\sum_{n=k}^{\infty} ar^n = \frac{a_1}{1-r}, |r| < 1$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} 1(x)^n \quad \text{or} \quad \boxed{\sum_{n=0}^{\infty} x^n}$$

$$a_1 = 1$$

$$r = x$$

**Find the Maclaurin Series for each of the following functions.**

3.  $\sin x^2$

\*look for parent function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$$

**Practice Problems:**

1. What is the coefficient of  $x^6$  in the Taylor Series about  $x = 0$  for the function  $f(x) = \frac{e^{2x^2}}{2}$  ?  $\frac{1}{2} e^{2x^2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^6}{6!}$$

$$e^{2x^2} = 1 + (2x^2) + \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3!}$$

$$\frac{1}{2} e^{2x^2} = \frac{1}{2} \left[ 1 + 2x^2 + \frac{4x^4}{2} + \frac{8x^6}{6} \right]$$

$$= \frac{1}{2} + x^2 + x^4 + \boxed{\frac{4}{6} x^6}$$

coefficient is  $\frac{2}{3}$

4.  $x^2 e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$x^2 e^x = x^2 \left[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \right]$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \dots + \frac{x^{n+2}}{n!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}}$$



2. If  $f(x) = x \sin 3x$ , what is the Taylor Series for  $f$  about  $x = 0$ ? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(3x) = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}$$

$$x \sin(3x) = x(3x) - \frac{x \cdot 27x^3}{6} + \frac{x \cdot 3^5 x^5}{5!} - \frac{x \cdot 3^7 x^7}{7!}$$

$$3x^2 - \frac{27x^4}{3!} + \frac{243x^6}{5!} - \frac{2187x^8}{7!}$$

3. What is the Maclaurin Series for  $\frac{1}{(1-x)^2}$ ? Write the first four non-zero terms.

$$\frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$[1 + x + x^2 + x^3 + x^4 + \dots] \cdot [1 + x + x^2 + x^3]$$

$$\begin{aligned} &1 + x + x^2 + x^3 \\ &+ x + x^2 + x^3 + x^4 \\ &+ x^2 + x^3 + x^4 + x^5 + \dots \\ &+ x^3 + x^4 + x^5 + \dots \\ &+ x^4 + x^5 + \dots \end{aligned}$$


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$$1 + 2x + 3x^2 + 4x^3$$

4. What is the Maclaurin Series for the function  $f(x) = \frac{1}{2}(e^x + e^{-x})$ ? Write the first four non-zero terms.

$$\frac{1}{2}e^x = \frac{1}{2} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \right)$$

$$\frac{1}{2}e^{-x} = \frac{1}{2} \left( 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} \right)$$

$$+ \frac{1}{2} \left[ 2 + 2\left(\frac{x^2}{2}\right) + 2\left(\frac{x^4}{4!}\right) + 2\left(\frac{x^6}{6!}\right) \right]$$

$$1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

5. Find the Maclaurin Series for the function  $f(x) = \cos \sqrt{x}$ . Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(\sqrt{x}) = 1 - \frac{\sqrt{x}^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!}$$

$$= 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!}$$

6. Find the Maclaurin Series for the function  $f(x) = \sin 5x$ . Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(5x) = 5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(5x)^7}{7!}$$

$$\sin(5x) = 5x - \frac{125x^3}{3!} + \frac{5^5 x^5}{5!} - \frac{5^7 x^7}{7!}$$

7. What is the Taylor series expansion about  $x = 0$  for the function  $f(x) = \frac{\sin x}{x}$ ? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\frac{1}{x} \cdot \sin x = \frac{1}{x} \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}$$

8. The sum of the series  $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \dots + \frac{3^n}{n!}$  is

$$e^x = \sum \frac{x^n}{n!} \quad e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

(A)  $\ln 3$

(B)  $e^3$

(C)  $\cos 3$

(D)  $\sin 3$

9. What is the sum of the series  $1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \dots + \frac{(\ln 3)^n}{n!}$ ?

$$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$e^{(\ln 3)} = 1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \dots + \frac{(\ln 3)^n}{n!}$$

$$e^{\ln 3} = 3$$

10. What is the Taylor Series about  $x = 0$  for the function  $f(x) = 1 + x^2 + \cos x$ ? Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$f(x) = 1 + x^2 + \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \right)$$

$$f(x) = 1 + x^2 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$f(x) = 2 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

11. What is the sum of the infinite series  $1 - \left(\frac{\pi}{2}\right)^2 \left(\frac{1}{3!}\right) + \left(\frac{\pi}{2}\right)^4 \left(\frac{1}{5!}\right) - \left(\frac{\pi}{2}\right)^6 \left(\frac{1}{7!}\right) + \dots + \frac{\left(\frac{\pi}{2}\right)^{2n} (-1)^n}{(2n+1)!}$  ?

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cdot x^1}{x \cdot (2n+1)!} \rightarrow \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\left. \begin{array}{l} \sin\left(\frac{\pi}{2}\right) \\ \frac{\pi}{2} \end{array} \right\} = \frac{1}{\pi/2} = \frac{2}{\pi}$$

12. Find the Maclaurin Series for the function  $f(x) = e^{-3x}$ . Write the first four non-zero terms.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6}$$

OR

$$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!}$$

$$\boxed{1 - 3x + \frac{9x^2}{2} - \frac{9}{2}x^3}$$

13. Find the Maclaurin Series for the function  $f(x) = \frac{\sin x^2}{x} + \cos x$ . Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!}$$

$$\frac{1}{x}(\sin(x^2)) = \frac{1}{x} \left[ x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \right]$$

$$= x - \frac{x^5}{3!} + \frac{x^9}{5!} - \frac{x^{13}}{7!}$$

$$\boxed{\frac{\sin x^2}{x} + \cos x \approx 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!}}$$

14. Which of the following is the Maclaurin Series for the function  $f(x) = x \cos 2x$ ?

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$x \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n} \cdot x}{(2n)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n} \cdot x^{2n+1}}{(2n)!}$$

(A)  $\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n}}{(2n)!}$       (B)  $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$       (C)  $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n)!}$       (D)  $\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+1}}{(2n)!}$

Finding Taylor or Maclaurin Series

15. The Maclaurin series  $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$  represents which function  $f(x)$

A)  $\sin x = x - \frac{x^3}{3!} + \dots$  (incorrect)

B)  $-\sin x = -(x - \frac{x^3}{3!} + \dots)$  (incorrect)

C)  $\frac{1}{2}e^x - \frac{1}{2}e^{-x} \rightarrow \frac{1}{2}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}) - \frac{1}{2}(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!})$

$x + \frac{1}{2}(\frac{2x^3}{3!}) + \frac{1}{2}(\frac{2x^5}{5!}) \rightarrow x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

(A)  $\sin x$

(B)  $-\sin x$

(C)  $\frac{1}{2}(e^x - e^{-x})$

(D)  $e^x - e^{-x}$

16. The function  $f$  satisfies the equation  $f'(x) = f(x) + x + 1$  and  $f(0) = 2$ . The Taylor Series for  $f$  about  $x = 0$  converges to  $f(x)$  for all  $x$ .

a. Write an equation for the line tangent to the curve of  $y = f(x)$  at  $x = 0$ .

$f'(x) = f(x) + x + 1$   
 $f'(0) = f(0) + 0 + 1$   
 $f'(0) = 2 + 1 = 3$

point:  $(0, 2)$   
slope:  $m = 3$

$y - 2 = 3(x - 0)$   
OR  
 $y = 3x + 2$

b. Find  $f''(0)$  and find the second-degree Taylor Polynomial for  $f$  about  $x = 0$ .

$f''(x) = f'(x) + 1$   
 $f''(0) = f'(0) + 1$   
 $= 3 + 1 = 4$

$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$   
 $P_2(x) = 2 + 3(x-0) + \frac{4}{2!}(x-0)^2$

$P_2(x) = 2 + 3x + 2x^2$

c. Find the fourth-degree Taylor Polynomial for  $f$  about  $x = 0$ .

$$f'''(x) = f''(x) \rightarrow f'''(0) = f''(0) = 4$$

$$f^4(x) = f'''(x) \rightarrow f^4(0) = f'''(0) = 4$$

$$P_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^4(0)}{4!}(x-0)^4$$

$$P_4(x) = 2 + 3(x-0) + \frac{4}{2!}(x-0)^2 + \frac{4}{3!}(x-0)^3 + \frac{4}{4!}(x-0)^4$$

$$P_4(x) = 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4$$

d. Find  $f^{(n)}(0)$ , the  $n^{\text{th}}$  derivative of  $f$  about  $x = 0$ , for  $n \geq 2$ . Use the Taylor Series for  $f$  about  $x = 0$  and the Taylor Series for  $e^x$  about  $x = 0$  to find  $f(x) - 4e^x$ .

$$f^{(n)}(0) = 4 \quad \text{for } n \geq 2$$

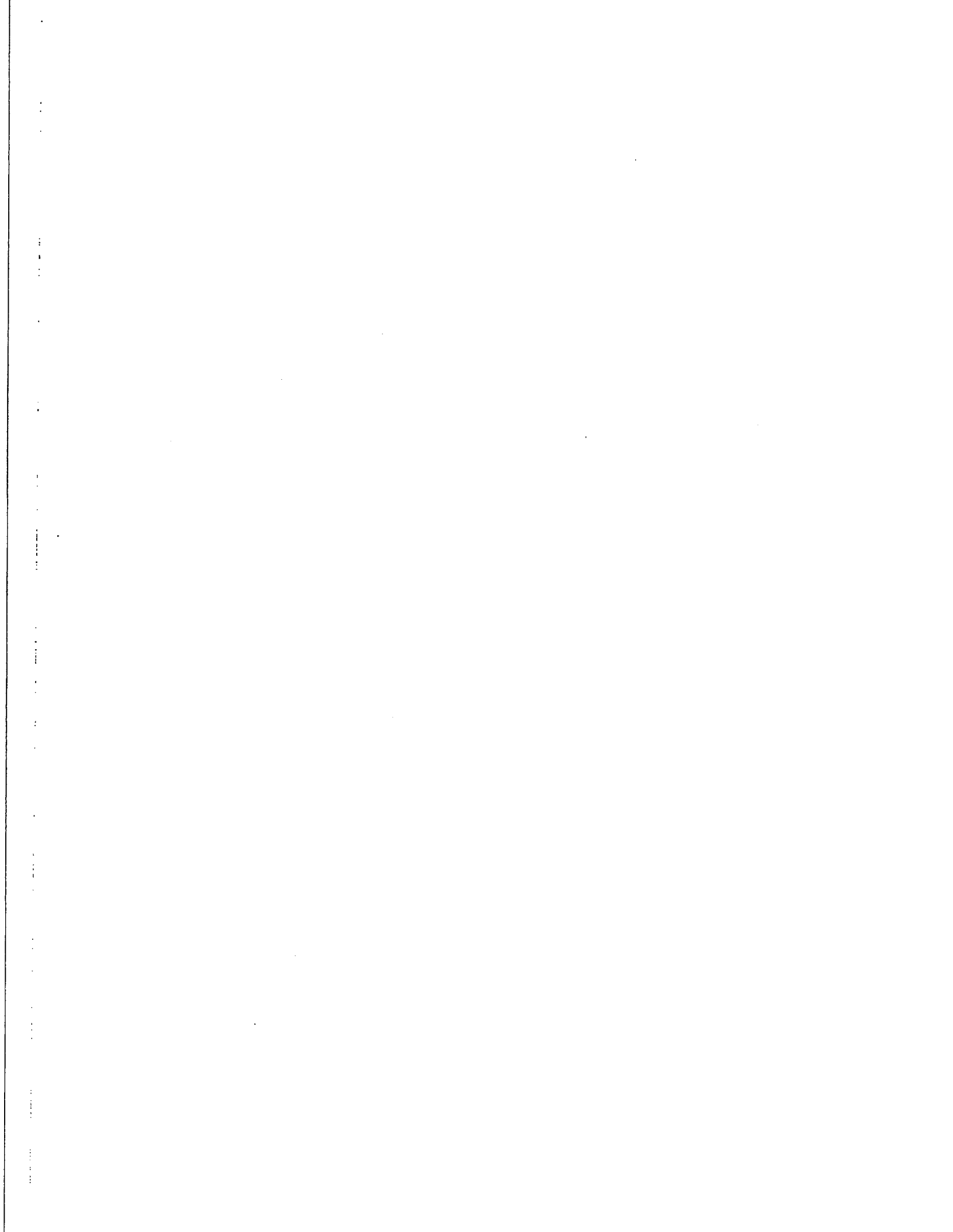
$$f(x) = 2 + 3x + \frac{4x^2}{2!} + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \dots$$

$$4e^x = 4\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)$$

$$= 4 + 4x + \frac{4x^2}{2!} + \frac{4x^3}{3!} + \frac{4x^4}{4!} + \dots$$

$$f(x) - 4e^x = 2 - 4 + 3x - 4x + \frac{4x^2}{2!} - \frac{4x^2}{2!} + \frac{4x^3}{3!} - \frac{4x^3}{3!} + \frac{4x^4}{4!} - \frac{4x^4}{4!} + \dots$$

$$f(x) - 4e^x = -2 - x$$



BC Calculus – 10.8b Notes – Representing Functions as a Power Series

Recall:

Function	Series (expanded)	Series Notation	Int. of Conv.
$e^x =$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin x =$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$\cos x =$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1+x} =$	$1 - x + x^2 - x^3 + \dots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$

1. If  $f(x) = \sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$  then  $f'(x) =$

$$f(x) = 1 + \frac{x^5}{1} + \frac{x^{10}}{2!} + \frac{x^{15}}{3!} + \dots$$

$$f'(x) = 0 + 5x^4 + \frac{10x^9}{2} + \frac{15x^{14}}{3!} + \dots$$

simplified

$$f'(x) = 5x^4 + 5x^9 + \frac{5x^{14}}{2} + \dots$$

OR take the derivative of the Rule

$$\sum_{n=1}^{\infty} \frac{5n \cdot x^{5n-1}}{n!} \rightarrow \frac{5n \cdot x^{5n-1}}{n(n-1)!}$$

$$\frac{5x^4}{1} + \frac{10x^9}{2!} + \dots$$

simplified  $\rightarrow$

$$\sum_{n=1}^{\infty} \frac{5x^{5n-1}}{(n-1)!}$$

2. Write the first 4 nonzero terms for the Maclaurin series that represents  $\int_0^x \sin(t^7) dt$ .

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(t^7) = t^7 - \frac{t^{21}}{3!} + \frac{t^{35}}{5!} - \frac{t^{49}}{7!} + \dots + \frac{(-1)^n (t^7)^{2n+1}}{(2n+1)!}$$

$$\int_0^x \sin(t^7) dt = \left[ \frac{t^8}{8} - \frac{t^{22}}{22 \cdot 3!} + \frac{t^{36}}{36 \cdot 5!} - \frac{t^{50}}{50 \cdot 7!} \right]_0^x = \frac{x^8}{8} - \frac{x^{22}}{22 \cdot 3!} + \frac{x^{36}}{36 \cdot 5!} - \frac{x^{50}}{50 \cdot 7!}$$

Practice Problems:

1. What is the coefficient of  $x^2$  in the Taylor Series for the function  $f(x) = \sin^2 x$  about  $x = 0$ ?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin^2 x = (\sin x)(\sin x) = \left[ x - \frac{x^3}{3!} + \dots \right] \left[ x - \frac{x^3}{3!} + \dots \right] \rightarrow x^2 - \frac{x^4}{3!} - \frac{x^4}{3!} + \dots$$

coefficient is  $\boxed{1}$

2. If the function  $f$  is defined as  $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$ , then what is  $f'(x)$ ? Write the first four nonzero terms and the general term.

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} (2n) x^{2n-1} = 2x + \frac{4}{2!} x^3 + \frac{6}{3!} x^5 + \frac{8}{4!} x^7$$

$$= \boxed{2x + 2x^3 + x^5 + \frac{1}{3}x^7}$$

3. Use the power series expansion for  $\cos x$  to evaluate the integral  $\int_0^x \cos t^6 dt$ . Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \rightarrow \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(t^6) = 1 - \frac{(t^6)^2}{2!} + \frac{(t^6)^4}{4!} - \frac{(t^6)^6}{6!} \rightarrow \frac{(-1)^n (t^6)^{2n}}{(2n)!}$$

$$\int_0^x \cos(t^6) dt = x - \frac{t^{13}}{26} + \frac{t^{25}}{25 \cdot 4!} - \frac{t^{37}}{37 \cdot 6!} \rightarrow \frac{(-1)^n t^{12n+1}}{(12n+1) \cdot (2n)!}$$

4. For  $x > 0$ , the power series defined by  $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots + \frac{(-1)^n x^{2n}}{(2n+1)!}$  converges to which of the following?

(A)  $\cos x$ (B)  $\sin x$ (C)  $\frac{\sin x}{x}$ (D)  $e^x - e^{-x}$ 

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{1}{x} \cdot \sin x = \frac{1}{x} \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]$$

$$= \left[ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \right]$$



5. It is known that the Maclaurin series for  $\frac{1}{1-x}$  is  $\sum_{n=0}^{\infty} x^n$ . Use this fact to assist in finding the first four nonzero terms and the general term for the power series expansion for the function  $\frac{x^2}{1-x^2}$ .

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

$$\frac{x^2}{1-x^2} = x^2 [1 + x^2 + x^4 + x^6 + \dots] \rightarrow x^2 + x^4 + x^6 + x^8 + \dots \quad \sum_{n=0}^{\infty} x^{2n+2}$$

6. Let  $f$  be the function with initial condition  $f(0) = 0$  and derivative  $f'(x) = \frac{1}{1+x^7}$ . Write the first four nonzero terms of the Maclaurin series for the function  $f$ .

$$\frac{1}{1+x} \rightarrow \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$\frac{1}{1+x^7} \rightarrow 1 - (x^7) + (x^7)^2 - (x^7)^3 + (x^7)^4 - \dots$$

$$1 - x^7 + x^{14} - x^{21} + x^{28} + \dots$$

$$\int \frac{1}{1+x^7} dx = x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + C$$

$C=0$   
since  $f(0)=0$

7. Find the Maclaurin series for the function  $f(x) = e^{3x}$ . Write the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$$e^{3x} \approx 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!}$$

8. If a function has the derivative  $f'(x) = \sin(x^2)$  and initial conditions  $f(0) = 0$ , write the first four nonzero terms of the Maclaurin series for  $f$ .

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$\int \sin(x^2) dx \approx \frac{x^3}{3!} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!}$$

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9. The function  $f$  has derivatives of all orders and the Maclaurin series for the function  $f$  is given by

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3}$ . Find the Maclaurin series for the derivative  $f'(x)$ . Write the first four nonzero terms and the general term.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \dots$$

$$f'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 - \frac{7}{9}x^6 + \dots + \frac{(-1)^n (2n+1)x^{2n}}{2n+3}$$

10. Let the function  $f$  be defined by  $f(x) = \frac{1}{1-x}$ . Find the Maclaurin series for the derivative  $f'$ . Write the first four nonzero terms and the general term.

$$f = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$f'(x) = 1 + 2x + 3x^2 + 4x^3 + \dots + n x^{n-1}$$

11. Find the second-degree Taylor Polynomial for the function  $f(x) = \frac{\cos x}{1-x}$  about  $x = 0$ .

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$(\cos x) \left( \frac{1}{1-x} \right) = \left[ 1 - \frac{x^2}{2!} + \dots \right] \left[ 1 + x + x^2 + \dots \right]$$

$$= 1 + x + x^2 - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!} + \dots$$

$$f(x) = 1 + x + x^2 - \frac{x^2}{2}$$

$$f(x) = 1 + x + \frac{x^2}{2}$$

12. What is the coefficient of  $x^2$  in the Maclaurin series for the function  $f(x) = \left( \frac{1}{1+x} \right)^2$ ?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \dots$$

$$\left( \frac{1}{1+x} \right)^2 = \left( \frac{1}{1+x} \right) \left( \frac{1}{1+x} \right) \rightarrow \left[ 1 - x + x^2 - \dots \right] \left[ 1 - x + x^2 - \dots \right]$$

$$= 1 - x + x^2 - x + x^2 - x^3 + x^2 - x^3 + x^4$$

$$= 1 - 2x + \underline{3x^2} - 2x^3 + x^4$$

Coefficient for  $x^2$  term  
is 3

13. Find the Maclaurin series for the function  $f(x) = x \cos x^2$ . Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots + \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$x \cdot \cos(x^2) = x \left[ 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots + \frac{(-1)^n x^{4n}}{(2n)!} \right] + \dots \rightarrow \frac{(-1)^n x^{4n+1}}{(2n)!}$$

14. Given that  $f$  is a function that has derivatives of all orders and  $f(1) = 3$ ,  $f'(1) = -2$ ,  $f''(1) = 2$ , and  $f'''(1) = 4$ . Write the second-degree Taylor Polynomial for the derivative  $f'$  about  $x = 1$  and use it to find the approximate value of  $f'(1.2)$ .

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = 3 + -2(x-1) + \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3$$

$$P_2'(x) = -2 + 2(x-1) + \frac{4}{3!}(3)(x-1)^2 = -2 + 2(x-1) + 2(x-1)^2$$

$$P_2'(1.2) \approx -2 + 2(0.2) + 2(0.2)^2 = \boxed{-1.52}$$

15. Let the fourth-degree Taylor Polynomial be defined by  $T = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$  for the function  $f$  about  $x = 4$ . Find the third-degree Taylor Polynomial for  $f'$  about  $x = 4$  and then use it to approximate  $f'(4.2)$ .

$$T_3'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$$

$$T_3'(4.2) = -3 + 10(0.2) - 6(0.2)^2 + 24(0.2)^3 \approx \boxed{-1.048}$$

### 16.5 Representing Functions as Power Series

### Test Prep

16. Given a function defined by  $f(x) = \frac{\cos(2x)-1}{x^2}$  for  $x \neq 0$  and is continuous for all real numbers  $x$ .

a. What is the limit of the function  $f(x)$  as  $x$  approaches 0?

$$\lim_{x \rightarrow 0} \frac{\cos(2x)-1}{x^2} \rightarrow \frac{1-1}{0} \rightarrow \frac{0}{0}$$

$$\text{L'Hopital's} \quad \lim_{x \rightarrow 0} \frac{-\sin(2x) \cdot 2}{2x} \rightarrow \frac{0}{0}$$

$$\text{L'Hopital's} \quad \lim_{x \rightarrow 0} \frac{-2\cos(2x) \cdot 2}{2} \rightarrow \frac{-4(1)}{2} = \boxed{-2}$$

- b. Write the first four nonzero terms and the general term of the power series that represents the function

$$h(x) = \cos 2x$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots + \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= 1 - \frac{4x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \dots + \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

- c. Use the results from part (b) to write the first three nonzero terms for  $f(x) = \frac{\cos(2x)-1}{x^2}$ .

$$\frac{1}{x^2} [\cos(2x) - 1]$$

$$-2 + \frac{2}{3}x^2 - \frac{4}{45}x^4$$

$$\frac{1}{x^2} \left[ 1 - 2x^2 + \frac{16}{4!}x^4 - \frac{4}{45}x^6 - 1 \right]$$

- d. Use the results from part (c) to determine if the function  $f(x) = \frac{\cos(2x)-1}{x^2}$  has a relative maximum, a relative minimum or neither at  $x = 0$ . Justify your answer.

$$f(x) \approx -2 + \frac{2}{3}x^2 - \frac{4}{45}x^4$$

$$f'(x) = \frac{2}{3} \cdot 2x - \frac{16}{45}x^3$$

$$0 = x \left( \frac{4}{3} - \frac{16}{45}x^2 \right)$$

$x=0$  is a critical point  
where slope = 0

$$f''(x) = \frac{4}{3} - \frac{48}{45}x^2$$

$$f''(0) = \frac{4}{3} - \frac{48}{45}(0)^2 = \frac{4}{3} > 0$$

so concave up at  $x=0$

$x=0$  is a relative minimum  
because  $f'(0)=0$  and  $f''(0) > 0$

Key

BC Calculus – 10.10b Notes – Lagrange Error Bound

Error bound means how far off is the approximation from the actual value or answer

Exact value = Approximate value + Remainder

$$f(x) = \underbrace{P(x)}_{\text{(Taylor polynomial)}} + \underbrace{R(x)}_{\text{(Remainder)}}$$

$$R(x) = f(x) - P(x)$$

$$\text{Error: } |R_n(x)| = |f(x) - P_n(x)|$$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots + \frac{f^{(n)}(c)(x-c)^n}{n!} + R(x)$$

$$\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!} \rightarrow \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$$

\* we want to know what is the greatest possible value of this (n+1) derivative on an interval

**Lagrange Error Bound**

Let  $f(x)$  be differentiable through the order  $n + 1$ . The error between the Taylor Polynomial and  $f(x)$  is bounded by:

$$|R_n(x)| \leq \frac{\text{Max}[f^{(n+1)}(z)](x-c)^{n+1}}{(n+1)!}$$

where  $z$  is some number between  $c$  and  $x$ .

centered at  $x=0$

1. The fourth degree Maclaurin polynomial for  $\cos x$  is given by  $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ . If this polynomial is used to approximate  $\cos(0.2)$ , what is the Lagrange error bound?

$$\cos(0.2) \approx P_4(0.2) = 0.980067$$

$x = 0.2$   
 $c = 0$   
 $n = 4$

$$R_4(x) \leq \frac{\text{Max}[-\sin(z)](0.2-0)^5}{(4+1)!}$$

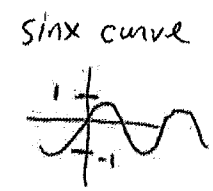
\*  $c \leq z \leq x$   
 $0 \leq z \leq 0.2$

- $f = \cos x$
- $f'(x) = -\sin x$
- $f''(x) = -\cos x$
- $f'''(x) = \sin x$
- $f^4(x) = \cos x$
- $f^5(x) = -\sin x$

$$R_4(x) \leq \frac{1 \cdot (0.2)^5}{5!}$$

$$R_4(x) \leq 2.667 \times 10^{-6}$$

$$R_4(x) \leq 0.000002667$$



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2. Use a third degree Taylor polynomial on the interval  $[0, 1]$  for  $e^x$  centered about  $x = 0$  to approximate  $e^1$ . What is the error bound of this approximation?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^1 \approx P_3(1) = 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} = 2.66667$$

$$R_3(x) \leq \left| \frac{\max [e^z] \cdot (1-0)^4}{4!} \right| \quad 0 \leq z \leq 1$$

$$R_3(x) \leq \left| \frac{e^1(1)^4}{4!} \right|$$

$$R_3 \leq 0.1132617$$

3. What is the smallest order Taylor Polynomial centered at  $x = 1$  which will approximate  $e^{x-1}$  on the interval  $[0, 3]$  with a Lagrange error bound less than 1?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{x-1} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots$$

$$* \quad 0 < z < 3$$

$$\left| \frac{\max (f^{(n+1)}(z)) \cdot (x-1)^{n+1}}{(n+1)!} \right| < 1$$

$$\underline{x=3} \quad \left| \frac{e^2 \cdot (3-1)^{n+1}}{(n+1)!} \right| < 1$$

$$\frac{e^2 \cdot 2^{n+1}}{(n+1)!} < 1$$

5<sup>th</sup> order Taylor Polynomial

n	R
1	14.778
2	9.8521
3	4.926
4	1.9704
5	0.6568

Practice Problems:

1. The third Maclaurin polynomial for  $\sin x$  is given by  $p(x) = x - \frac{x^3}{3!}$ . If this polynomial is used to approximate  $\sin(0.1)$ , what is the Lagrange error bound?

$$\text{Interval } [0, 0.1]$$

$$\frac{\max [f^{(n+1)}(z)] [x-c]^{n+1}}{(n+1)!}$$

$$\frac{[1][0.1]^4}{4!} \rightarrow 4.1667 \times 10^{-6}$$

$$\frac{\max [f^4(z)] [0.1-0]^4}{4!}$$

$$f = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^4(x) = \sin x$$

max of  $\sin x$  is 1

2. If the Taylor Polynomial for approximating  $\cos x$  is given by  $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ , what is the upper bound for the error in the approximation of  $\cos(0.3)$ ?

Interval  $[0, 0.3]$   
 $n=4$   
 $\frac{\text{Max}[f^5(z)] [0.3-0]^5}{5!}$

$f = \cos x$   
 $f' = -\sin x$   
 $f'' = -\cos x$   
 $f''' = \sin x$   
 $f^4 = \cos x$   
 $f^5 = -\sin x$   
 Max of  $-\sin x = 1$

$\frac{(1)(0.3)^5}{5!} = 2.025 \times 10^{-5}$

3. If the Taylor Polynomial about  $x = 0$  for the approximation of  $e^x$  is given by  $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$ , what is the upper bound for the error in the approximation of  $e$ ?

$f^6(x) = e^x$  Max on interval  $[0, 1]$  is  $e$  →  $e$  means  $e^1$  → interval  $[0, 1]$

$\frac{\text{Max}[f^6(z)] [1-0]^6}{6!} \rightarrow \frac{(e)(1)^6}{6!} = 0.00377$

4. Let  $f$  be a function that has derivatives of all orders for all real numbers and let  $P_3(x)$  be the third-degree Taylor Polynomial for  $f$  about  $x = 0$ .  $|f^{(n)}(x)| \leq \frac{n}{n+1}$ , for  $1 \leq n \leq 5$  and all values of  $x$ . Of the following, which is the smallest value of  $k$  for which the Lagrange error bound guarantees that  $|f(1) - P_3(1)| \leq k$ ?

$R_3(x) \leq k$  Max  $[f^4(z)] \leq \frac{4}{4+1} = \frac{4}{5}$  on interval  $[0, 1]$

$\frac{\text{Max}[f^4(z)] [x-c]^4}{4!} \rightarrow \frac{(\frac{4}{5})(1-0)^4}{4!} \rightarrow \frac{4}{5} \cdot \frac{1}{4!}$

- (A)  $\frac{5}{6}$  (B)  $\frac{5}{6} * \frac{1}{5!}$  (C)  $\frac{5}{6} * \frac{1}{4!}$  (D)  $\frac{4}{5} * \frac{1}{4!}$

5. The function  $f$  has derivatives of all orders for all real numbers,  $f^{(4)}(x) = e^{\cos x}$ . If the third-degree Taylor Polynomial for  $f$  about  $x = 0$  is used to approximate  $f$  on the interval  $[0, 1]$ , what is the Lagrange error bound?

$\frac{\text{Max}[f^4(z)] [1-0]^4}{4!} \rightarrow \frac{(e^1)(1)^4}{4!} \approx 0.11326$

Max of  $e^{\cos x}$  on  $[0, 1]$  is  $x=0 \rightarrow e^{\cos 0} \rightarrow e^1$

6. The Taylor series for a function  $f$  about  $x = 3$  is given by  $\sum_{n=0}^{\infty} (-1)^n \frac{3n+1}{2^n} (x-3)^n$  and converges to  $f$  for  $0 \leq x \leq 5$ . If the third-degree Taylor Polynomial for  $f$  about  $x = 3$  is used to approximate  $f\left(\frac{13}{4}\right)$ , what is the alternating series error bound?

$$a_{4} = \frac{3(4)+1}{2^4} (x-3)^4 \rightarrow \frac{13}{16} \left[\frac{13}{4}-3\right]^4$$

$$|R_3| \leq \frac{13}{16} \left(\frac{1}{4}\right)^4$$

7. Let  $f$  be a polynomial function with nonzero coefficients such that  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ .  $T_4(x)$  is the fourth-degree Taylor Polynomial for  $f$  about  $x = c$  such that  $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$ . Based on the Lagrange error bound,  $f(x) - T_4(x)$  must equal which of the following?

$$|R_4(x)| = |f(x) - T_4(x)| \text{ is the next term}$$

(A)  $x$

(B)  $(x-c)^5$

(C)  $a_5(x-c)^5$

(D)  $\frac{a_5(x-c)^5}{5!}$

8. Let  $P(x)$  be the sixth-degree Taylor Polynomial for a function  $f$  about  $x = 0$ . Information about the maximum of the absolute value of selected derivatives of  $f$  over the interval  $0 \leq x \leq 1.5$  is given below.

$$\max_{0 \leq x \leq 1.5} |f^{(5)}(x)| = 9.3$$

$$\max_{0 \leq x \leq 1.5} |f^{(6)}(x)| = 62.1$$

$$\max_{0 \leq x \leq 1.5} |f^{(7)}(x)| = 481.3$$

What is the smallest value of  $k$  for which the Lagrange error bound guarantees that  $|f(1.5) - P(1.5)| \leq k$ ?

$$\frac{\max |f^{(7)}(z)| [x-c]^7}{7!} \rightarrow \frac{(481.3)(1.5-0)^7}{7!} \approx 1.6316$$

on interval  $[0, 1.5]$

9. The function  $f$  has derivatives of all orders for all real numbers. Values of  $f$  and its first four derivatives at  $x = 2$  are given in the table.

$x$	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
2	6	-12	18	-24	34

- a. Write the third-degree Taylor Polynomial for  $f$  about  $x = 2$ , and use it to approximate  $f(1.5)$ .

$$P_3(x) = f(x) + f'(x)[x-2] + \frac{f''(x)}{2!}(x-2)^2 + \frac{f'''(x)}{3!}(x-2)^3$$

$$P_3(x) = 6 + -12(x-2) + \frac{18}{2}(x-2)^2 - \frac{24}{3!}(x-2)^3$$

$$P_3(x) = 6 - 12(x-2) + 9(x-2)^2 - 4(x-2)^3$$

$$f(1.5) \approx P_3(1.5) = 6 - 12(-0.5) + 9(-0.5)^2 - 4(-0.5)^3 = 14.75$$



b. The fourth derivative of  $f$  satisfies the inequality  $|f^{(4)}(x)| \leq 48$  for all  $x > 1$ . Use the Lagrange error bound to show that the approximation found in part (a) differs from  $f(1.5)$  by no more than  $\frac{1}{8}$ .

$$R_3(1.5) \leq \frac{\max [f^{(4)}(z)] (1.5-2)^4}{4!} = \frac{48(0.5)^4}{4!} = 0.125$$

$$R_3(1.5) \leq 0.125 = \frac{1}{8}$$

10. Let  $h$  be a function having derivatives of all orders for  $x > 0$ . Selected values for the first four derivatives of  $h$  are given for  $x = 3$ . Use the Lagrange error bound to show that the third-degree Taylor polynomial for  $h$  about  $x = 3$  approximates  $h(2.9)$  with an error less than  $3 \times 10^{-4}$ .

$x$	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	$\frac{3483}{8}$	$\frac{1125}{16}$

interval  
[2.9, 3]  
x, c

$$R_3(x) \leq \frac{\max [h^{(4)}(z)] (x-c)^4}{4!}$$

$$\leq \frac{(\frac{1125}{16})(2.9-3)^4}{4!}$$

$$\leq 2.9297 \times 10^{-4}$$

10.12 Lagrange Error Bound

Test Prep

11. Calculator allowed.

$x$	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function  $f$  has derivatives of all orders for all real numbers. Values of  $f$  and its first four derivatives at  $x = 3$  are given in the table.

a. Write an equation for the line tangent to the graph of  $f$  at  $x = 3$  and use it to approximate  $f(2.5)$ .

point: (3, 4)

slope:  $f'(3) = -8$

$$y - 4 = -8(x - 3)$$

$$y = -8(x - 3) + 4$$

$$y(2.5) = -8(2.5) + 28 = 8$$

$$f(2.5) \approx 8$$

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$c = 3$

- b. Write the third-degree Taylor polynomial for  $f$  about  $x = 3$ , and use it to approximate  $f(2.5)$ .

$$P_3(x) = f(x) + f'(x)(x-3) + \frac{f''(x)}{2!}(x-3)^2 + \frac{f'''(x)}{3!}(x-3)^3$$

$$P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$$

$$P_3(x) = 4 - 8(x-3) + \frac{14}{2}(x-3)^2 - \frac{22}{3!}(x-3)^3$$

$$f(2.5) \approx P_3(2.5) = 4 - 8(-0.5) + 7(-0.5)^2 - \frac{11}{3}(-0.5)^3 = \boxed{10.208}$$

- c. Is there enough information to determine whether  $f$  has a critical point at  $x = 2.5$ ? If not, explain why not. If so, determine whether  $f(2.5)$  is a relative maximum, relative minimum, or neither, and give a reason for your answer.

There is not enough information. We don't know if  $f'(2.5) = 0$  or if  $f'(2.5)$  does not exist.

The Taylor Polynomial only gives us an approximation of  $f(x)$ .

- d. The fourth derivative of  $f$  satisfies the inequality  $|f^{(4)}(x)| \leq 48$  for all  $x > 2$ . Use the Lagrange error bound to show that the approximation found in part (b) differs from  $f(2.5)$  by no more than  $\frac{1}{8}$ .

$$R_3(x) \leq \frac{\max [f^{(4)}(z)] [x-c]^4}{4!} \quad \text{interval } [2.5, 3]$$

$$\leq \frac{(48)(2.5-3)^4}{4!}$$

$$\leq \boxed{0.125}$$

- e. What is the coefficient of the  $(x-3)^3$  term in the Taylor series for  $f'$ , the derivative of  $f$ , about  $x = 3$ ?

$$f(x) = \dots \frac{f^{(4)}(x)}{4!}(x-3)^4$$

$$= \frac{30}{4!}(x-3)^4$$

$$f'(x) = \dots \frac{4 \cdot 30}{4!}(x-3)^3$$

coefficient is 5

$$\frac{30}{3!}(x-3)^3 \quad \text{or} \quad \frac{30}{6}(x-3)^3 \rightarrow \boxed{5}(x-3)^3$$

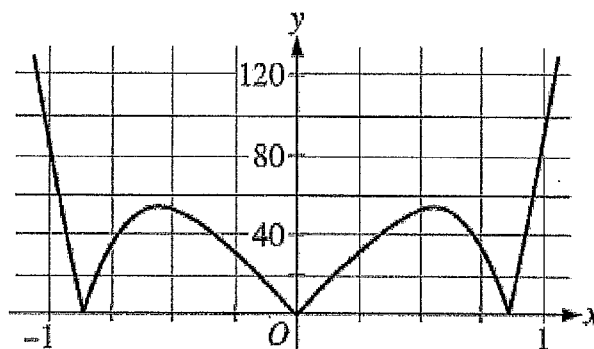
(48)

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**Question 6**

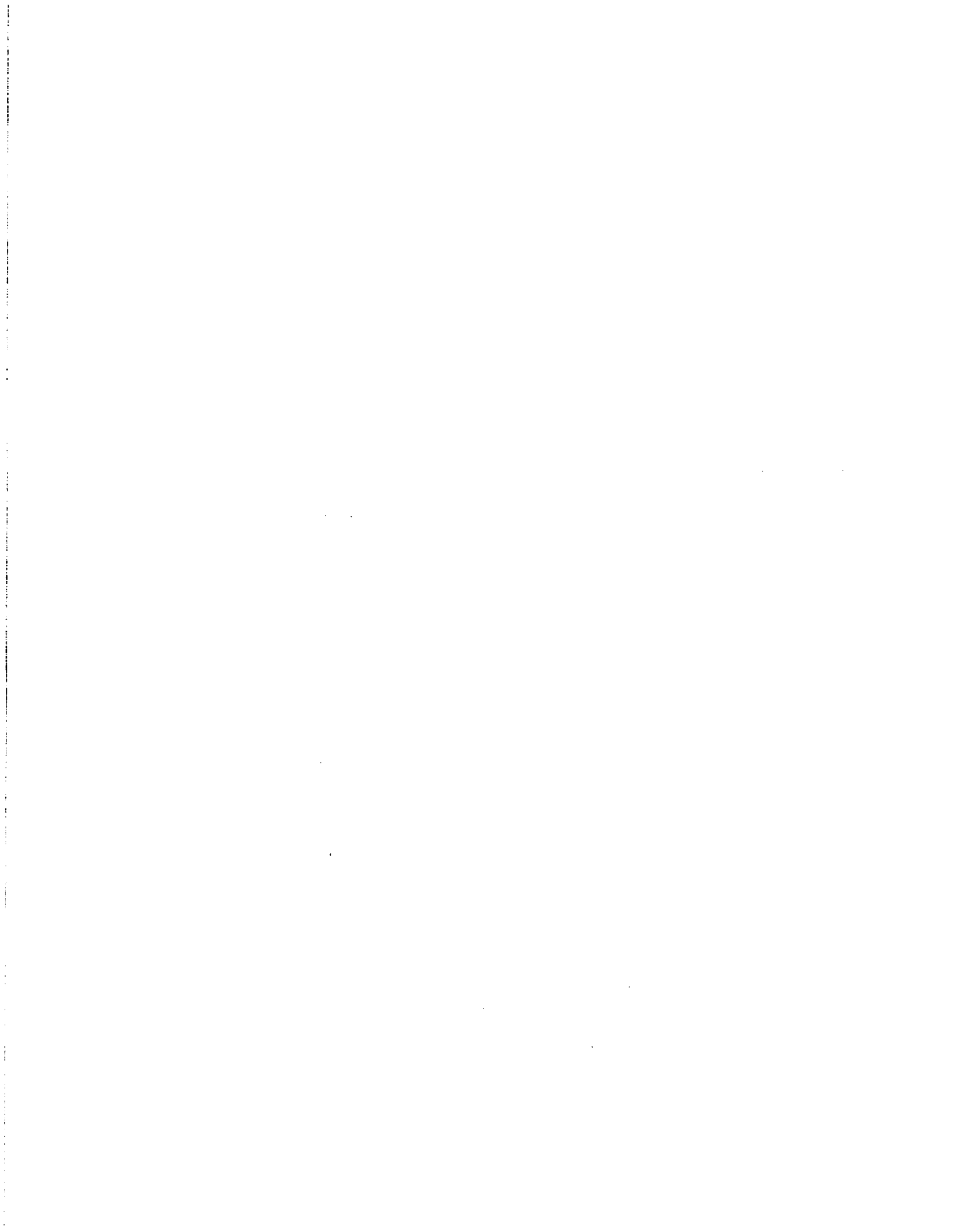
Let  $f(x) = \sin(x^2) + \cos x$ . The graph of  $y = |f^{(5)}(x)|$  is shown above.

- (a) Write the first four nonzero terms of the Taylor series for  $\sin x$  about  $x = 0$ , and write the first four nonzero terms of the Taylor series for  $\sin(x^2)$  about  $x = 0$ .



Graph of  $y = |f^{(5)}(x)|$

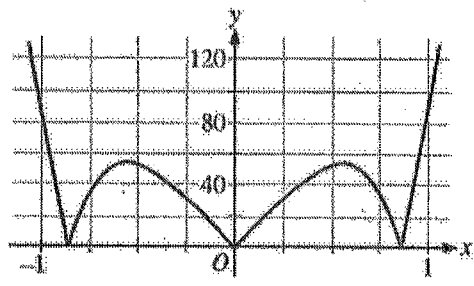
- (b) Write the first four nonzero terms of the Taylor series for  $\cos x$  about  $x = 0$ . Use this series and the series for  $\sin(x^2)$ , found in part (a), to write the first four nonzero terms of the Taylor series for  $f$  about  $x = 0$ .
- (c) Find the value of  $f^{(6)}(0)$ .
- (d) Let  $P_4(x)$  be the fourth-degree Taylor polynomial for  $f$  about  $x = 0$ . Using information from the graph of  $y = |f^{(5)}(x)|$  shown above, show that  $\left| P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right) \right| < \frac{1}{3000}$ .



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Question 6

Let  $f(x) = \sin(x^2) + \cos x$ . The graph of  $y = |f^{(5)}(x)|$  is shown above.



Graph of  $y = |f^{(5)}(x)|$

- (a) Write the first four nonzero terms of the Taylor series for  $\sin x$  about  $x = 0$ , and write the first four nonzero terms of the Taylor series for  $\sin(x^2)$  about  $x = 0$ .
- (b) Write the first four nonzero terms of the Taylor series for  $\cos x$  about  $x = 0$ . Use this series and the series for  $\sin(x^2)$ , found in part (a), to write the first four nonzero terms of the Taylor series for  $f$  about  $x = 0$ .
- (c) Find the value of  $f^{(6)}(0)$ .
- (d) Let  $P_4(x)$  be the fourth-degree Taylor polynomial for  $f$  about  $x = 0$ . Using information from the graph of  $y = |f^{(5)}(x)|$  shown above, show that  $|P_4(\frac{1}{4}) - f(\frac{1}{4})| < \frac{1}{3000}$ .

\*  $R_n(x) = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1} \right|$

a)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$   
 $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

b)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

c) Since  $\frac{f^{(6)}(0)}{6!} x^6 = -\frac{121}{6!} x^6$ ,

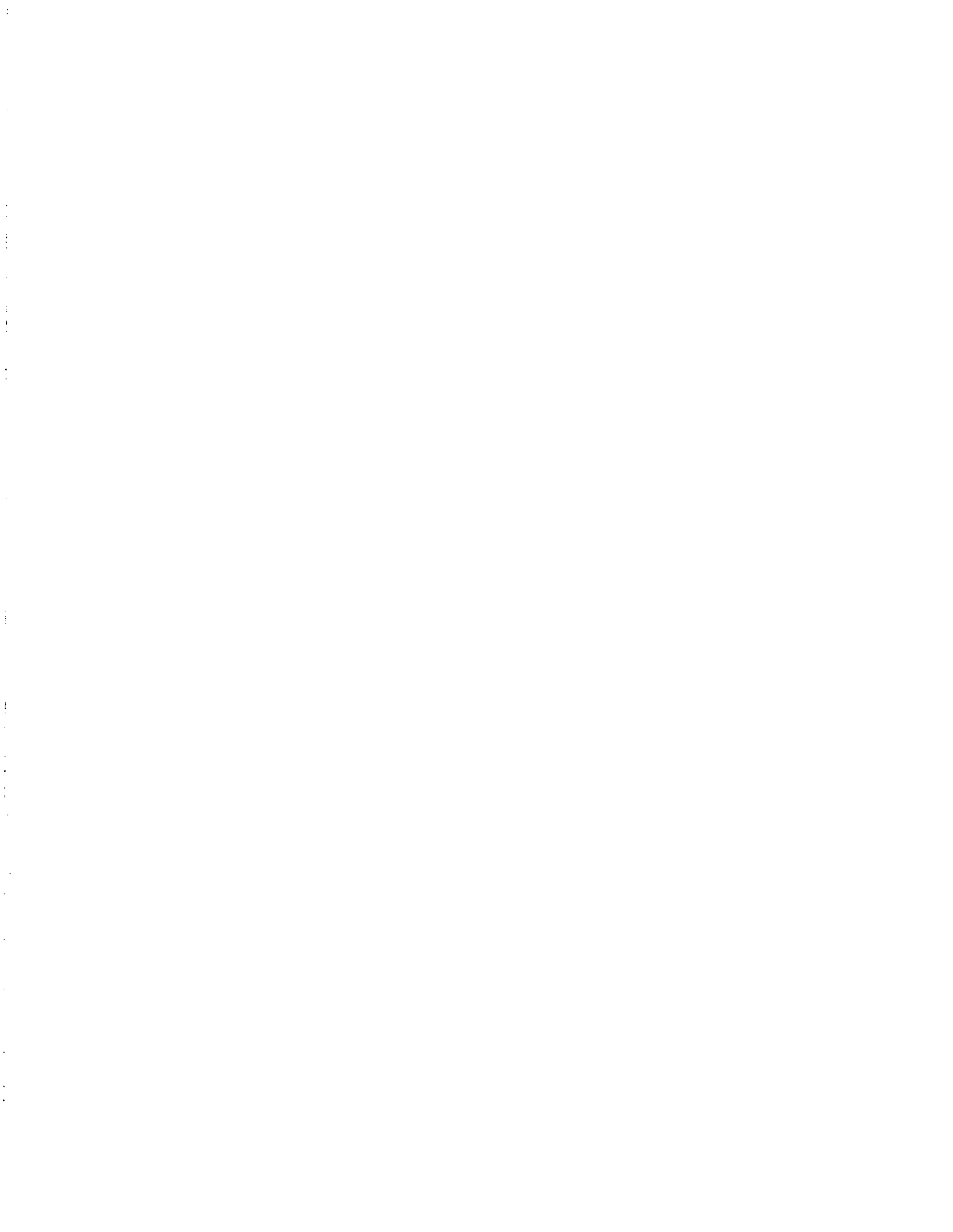
$f^{(6)}(0) = -121$

$f(x) = 1 + x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} - \frac{x^6}{6!}$   
 $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6} - \frac{x^6}{720}$   
 $1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{120x^6}{720} - \frac{1x^6}{720}$   
 $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121}{6!} x^6$

d)  $R_4(x) = |P_4(x) - f(x)| = |P_4(\frac{1}{4}) - f(\frac{1}{4})| = \left| \frac{f^{(5)}(z)}{5!} (x-c)^5 \right| \leq \left| \frac{40}{5!} (\frac{1}{4}-0)^5 \right|$

$= \left| \frac{40}{5!} (\frac{1}{4})^5 \right| = \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$   
 $= \frac{1}{3 \cdot 4^5} = \frac{1}{3 \cdot 1024}$

$x = \frac{1}{4}$   
 $c = 0$   
 max y-value in interval  $0 < x < \frac{1}{4}$

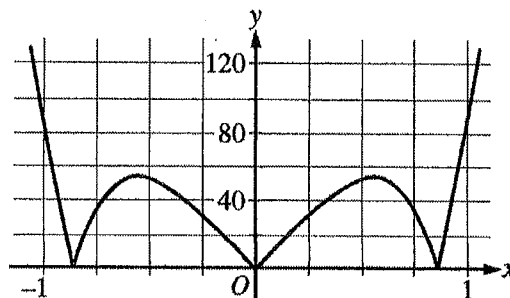


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Question 6

Let  $f(x) = \sin(x^2) + \cos x$ . The graph of  $y = |f^{(5)}(x)|$  is shown above.



Graph of  $y = |f^{(5)}(x)|$

- (a) Write the first four nonzero terms of the Taylor series for  $\sin x$  about  $x = 0$ , and write the first four nonzero terms of the Taylor series for  $\sin(x^2)$  about  $x = 0$ .
- (b) Write the first four nonzero terms of the Taylor series for  $\cos x$  about  $x = 0$ . Use this series and the series for  $\sin(x^2)$ , found in part (a), to write the first four nonzero terms of the Taylor series for  $f$  about  $x = 0$ .
- (c) Find the value of  $f^{(6)}(0)$ .
- (d) Let  $P_4(x)$  be the fourth-degree Taylor polynomial for  $f$  about  $x = 0$ . Using information from the graph of  $y = |f^{(5)}(x)|$  shown above, show that  $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$ .

(a)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$   
 $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$

3 :  $\begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin(x^2) \end{cases}$

(b)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$   
 $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \dots$

3 :  $\begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c)  $\frac{f^{(6)}(0)}{6!}$  is the coefficient of  $x^6$  in the Taylor series for  $f$  about  $x = 0$ . Therefore  $f^{(6)}(0) = -121$ .

1 : answer

(d) The graph of  $y = |f^{(5)}(x)|$  indicates that  $\max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)| < 40$ .

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \leq \frac{\max_{0 \leq x \leq \frac{1}{4}} |f^{(5)}(x)|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$$

2 :  $\begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

